PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 6, June 1995

MAYER-VIETORIS FORMULA FOR THE DETERMINANT OF A LAPLACE OPERATOR ON AN EVEN-DIMENSIONAL MANIFOLD

YOONWEON LEE

(Communicated by Peter Li)

ABSTRACT. Let Δ be a Laplace operator acting on differential p-forms on an even-dimensional manifold M. Let Γ be a submanifold of codimension 1. We show that if B is a Dirichlet boundary condition and R is a Dirichlet-Neumann operator on Γ, then $\operatorname{Det}(\Delta+\lambda) = \operatorname{Det}(\Delta+\lambda, B)\operatorname{Det}(R+\lambda)$ and $\operatorname{Det}^*\Delta = \frac{1}{(\det A)^2}\operatorname{Det}(\Delta, B)\operatorname{Det}^* R$. This result was established in 1992 by Burghelea, Friedlander, and Kappeler for a 2-dimensional manifold with $p = 0$.

1. INTRODUCTION

Let M be a compact oriented Riemannian manifold of dimension d, and let Γ be a submanifold of M with dimension $d-1$ such that Γ has a collared neighborhood U diffeomorphic to $\Gamma\times(-1, 1)$. Let M_Γ be the compact manifold with boundary $\Gamma\cup\Gamma$ obtained by cutting M along Γ. Let $E=\Lambda^p T^*M$ be a p-th exterior product of the cotangent bundle T^*M, $i\colon M_\Gamma\to M$ be the identification map, and $E_\Gamma := i^*E$.

Define the Dirichlet boundary condition $(\Delta+\lambda, B)$ to be

$$(\Delta+\lambda, B)\colon C^\infty(M_\Gamma, E_\Gamma)\to C^\infty(M_\Gamma, E_\Gamma)\oplus C^\infty(\partial M_\Gamma, E_\Gamma|_{\partial M_\Gamma}),$$
$$\omega\mapsto((\Delta+\lambda)\omega, \omega|_{\partial M_\Gamma}).$$

Define the Poisson operator P_B to be the restriction of $(\Delta+\lambda, B)^{-1}$ to $0\oplus C^\infty(\partial M_\Gamma, E_\Gamma|_{\partial M_\Gamma})$. Let ν be a unit normal vector field along ∂M_Γ; one can extend ν to be a global vector field on M_Γ by using a cut-off function. Define the Neumann boundary condition C to be

$$C\colon C^\infty(M_\Gamma, E_\Gamma)\to C^\infty(\partial M_\Gamma, E_\Gamma|_{\partial M_\Gamma}),$$
$$\omega\mapsto\nabla_\nu\omega|_{\partial M_\Gamma}.$$

Received by the editors September 13, 1993.
1991 *Mathematics Subject Classification*. Primary 58G26.

Definition. For any positive real number $\lambda > 0$, define $R(\lambda)$ to be the composition of the following maps:

$$\begin{aligned} C^\infty(\Gamma, E|_\Gamma) &\xrightarrow{\Delta_{ia}} C^\infty(\Gamma, E|_\Gamma) \oplus C^\infty(\Gamma, E|_\Gamma) \oplus 0 \\ &\xrightarrow{P_B} C^\infty(M_\Gamma, E_\Gamma) \\ &\xrightarrow{C} C^\infty(\Gamma, E|_\Gamma) \oplus C^\infty(\Gamma, E|_\Gamma) \\ &\xrightarrow{-\Delta_{if}} C^\infty(\Gamma, E|_\Gamma), \end{aligned}$$

where Δ_{ia} is the diagonal inclusion and Δ_{if} is the difference map.

Then $R(\lambda)$ is a positive definite selfadjoint elliptic operator. When $\lambda = 0$, both the Laplacian Δ and R have zero eigenvalues and so $\det\Delta = \det R = 0$. In this case we define the modified determinants $\det^*\Delta$ and $\det^* R$ to be the determinants of Δ and R respectively, when restricted to the orthogonal complement of the null space.

In [BFK], Burghelea, Friedlander, and Kappeler proved that on a 2-dimensional manifold and for the trivial line bundle $E = \Lambda^0 T^*M$,

(1) $\mathrm{Det}(\Delta + \lambda) = \mathrm{Det}(\Delta + \lambda, B)\,\mathrm{Det}\,R(\lambda)$ for $\lambda > 0$,
(2) $\mathrm{Det}^*\Delta = \frac{V}{l}\,\mathrm{Det}(\Delta, B)\,\mathrm{Det}^* R$,

where V is the area of the manifold and l is the length of Γ.

Let $\mathscr{H}_p$ be the space of harmonic p-forms equipped with the natural inner product $\langle\varphi, \psi\rangle = \int_M \varphi \wedge *\psi = \int_M (\varphi, \psi) d\mathrm{vol}(M)$, where $(\ ,\ )$ is a metric in $E = \Lambda^p T^*M$ induced by the Riemannian metric g on M. Let $\mathscr{H}_p|_\Gamma$ be the restriction of harmonic p-forms to Γ. Define an inner product on $\mathscr{H}_p|_\Gamma$ by $\langle\alpha, \beta\rangle_\Gamma = \int_\Gamma (\alpha, \beta)\, d\mu_\Gamma$, where $d\mu_\Gamma$ is a volume element of Γ coming from g restricted to Γ.

Suppose $k = \dim \mathscr{H}_p$, and let $\psi_1, \dots, \psi_k$ be an orthonormal basis of $\mathscr{H}_p$ and $\phi_1, \dots, \phi_k$ be an orthonormal basis of $\mathscr{H}_p|_\Gamma$. Let $J\colon \mathscr{H}_p \to \mathscr{H}_p|_\Gamma$ denote the restriction map. Let $J(\psi_i) = a_{ij}\phi_j$ and let $A = (a_{ij})_{1\le i, j\le k}$. In this paper we extend the result of Burghelea et al. to arbitrary even dimensions and arbitrary p-forms.

If M is a compact oriented Riemannian manifold of dimension d with d even and $E = \Lambda^p T^*M$, then

Theorem A. $\mathrm{Det}(\Delta + \lambda, B) = \mathrm{Det}(\Delta + \lambda, B)\,\mathrm{Det}\,R(\lambda)$ *for any* $\lambda > 0$.

Theorem B. $\mathrm{Det}^*\Delta = \frac{1}{(\det A)^2}\,\mathrm{Det}(\Delta, B)\,\mathrm{Det}^* R$.

Remark. If $p = 0$, then $E = M \times R$, and the matrix A is $(\frac{\sqrt{l}}{\sqrt{V}})$. Hence Theorem B reduces to

$$\mathrm{Det}^*\Delta = \frac{V}{l}\,\mathrm{Det}(\Delta, B)\,\mathrm{Det}^* R,$$

as stated in [BFK].

II. The proof of Theorem A

In [BFK], it is shown that

$$\mathrm{Det}(\Delta + \lambda) = c\,\mathrm{Det}(\Delta + \lambda, B) \cdot \mathrm{Det}\,R(\lambda),$$

and that $\log\operatorname{Det}(\Delta+\lambda)$, $\log\operatorname{Det}(\Delta+\lambda, B)$, and $\log\operatorname{Det} R(\lambda)$ admit asymptotic expansions;

$$\log\operatorname{Det}(\Delta+\lambda), \log\operatorname{Det}(\Delta+\lambda, B) \sim \sum_{k=-d}^{\infty} \alpha_k|\lambda|^{-k/2} + \beta_0 \log|\lambda|, \qquad \text{with } \alpha_0 = 0,$$

$$\log\operatorname{Det} R(\lambda) \sim \sum_{j=-d}^{\infty} \pi_j|\lambda|^{-j/2} + \sum_{j=0}^{d} q_j|\lambda|^{j/2}\log|\lambda|, \qquad \text{with}$$

$$\pi_0 = \sum_j \frac{\partial}{\partial s}\frac{1}{(2\pi)^{d-1}}\int_{R^{d-1}} J_{d-1}(s, \lambda; x)\varphi_j(x)|_{s=0}\, d\operatorname{vol}(x),$$

where

$$J_{d-1} = (s, \lambda; x) = \frac{1}{2\pi i}\int_{R^{d-1}} d\xi \int_\gamma \mu^{-s} r_{-1-(d-1)}\left(\mu, \frac{\lambda}{|\lambda|}, x, \xi\right) d\mu,$$

$$r_{-1} = (\mu - p_1(\lambda, x, \xi))^{-1},$$

$$r_{-1-j} = -(\mu - p_1(\lambda, x, \xi))^{-1} \cdot \sum_{k=0}^{j-1} \sum_{\substack{\alpha \\ |\alpha|+l=j-k}} \frac{1}{\alpha!}\partial_\xi^\alpha p_{1-l}(\lambda, x, \xi) D_x^\alpha r_{-1-k}(\mu, \lambda, x, \xi)$$

$\sigma(R(\lambda)) \sim p_1 + p_0 + p_{-1} + \cdots$ asymptotic symbol of $R(\lambda)$, $\{\varphi_j\}$ a partition of unity subordinate to coordinate charts, and γ is a curve on a complex plane enclosing all the eigenvalues of $R(\lambda)$ counterclockwise.

Hence

$$\log c = -\pi_0.$$

The proof of Theorem A reduces to the verification of the following equation:

$$p_{1-j}(x, -\xi, \lambda) = (-1)^j p_{1-j}(x, \xi, \lambda).$$

Then $r_{-1-j}(\mu, \frac{\lambda}{|\lambda|}, x, -\xi) = (-1)^j r_{-1-j}(\mu, \frac{\lambda}{|\lambda|}, x, \xi)$, so when d is even, $r_{-1-(d-1)}(\mu, \frac{\lambda}{|\lambda|}, x, \xi)$ is odd with respect to ξ. So $J_{d-1} = 0$ and $\pi_0 = 0$. Therefore we conclude

$$\operatorname{Det}(\Delta+\lambda) = \operatorname{Det}(\Delta+\lambda, B)\operatorname{Det} R(\lambda).$$

Definition. Let U be a collared neighborhood of Γ diffeomorphic to $\Gamma \times (-1, 1)$ with diffeomorphism $\eta\colon U \to \Gamma\times(-1, 1)$. Let $\Gamma_t = \eta^{-1}(\Gamma\times t)$, $-1 < t < 1$. Let N_t^+, N_t^- be Neumann operators to each side with respect to $\Delta+\lambda$; i.e. if $\varphi \in C^\infty(\Gamma_t, E|_{\Gamma_t})$, define $N_t^+(\varphi) = \nabla_{\nu_t} u|_{\Gamma_t}$, where $(\Delta+\lambda)u = 0$ in $M - \Gamma_t$, $u|_{\Gamma_t} = \varphi$, and ν_t is a normal vector field along Γ_t.

Then

$$R(\lambda) = -(N_0^+ + N_0^-).$$

Lemma 1. *In a local coordinate system such that the first fundamental form looks like*

$$\begin{pmatrix} g_{ij}(x, t) & 0 \\ 0 & 1 \end{pmatrix}$$

on $\Gamma \times (-1, 1)$, the Laplacian is $\Delta = -\frac{d^2}{dt^2} + F(x, t)\frac{d}{dt} + \Delta_t$, where Δ_t is the Laplacian on Γ_t and $F(x, t)$ is a C^∞-function valued $\binom{d}{p} \times \binom{d}{p}$ matrix. Then

$$\frac{dN_t^+}{dt} = -(N_t^+)^2 + F(x, t)N_t^+ + (\Delta_t + \lambda),$$
$$\frac{dN_t^-}{dt} = (N_t^-)^2 + F(x, t)N_t^- - (\Delta_t + \lambda).$$

Remark. The idea to consider the Neumann operator as a solution of operator-valued differential equations goes back to I. M. Gel′fand.

Proof. It is enough to show that the first statement is true. Let $\varphi \in C^\infty(\Gamma_t, E|_{\Gamma_t})$. Choose $u(x, t) \in C^\infty(M_{\Gamma_t}, E_{\Gamma_t})$ such that $(\Delta + \lambda)u(x, t) = 0$ on $M - \Gamma_t$ and $u(x, t)|_{\Gamma_t} = \varphi$. Then

$$\frac{d}{dt}u(x, t) = N_t^+(u(x, t)),$$
$$\frac{d^2}{dt^2}u(x, t) = \frac{d}{dt}(N_t^+(u(x, t))) = \frac{dN_t^+}{dt}(u(x, t)) + N_t^+\left(\frac{du}{dt}\right)$$
$$= \left(\frac{dN_t^+}{dt} + (N_t^+)^2\right)u(x, t), \quad \text{and}$$
$$\frac{d^2}{dt^2}u(x, t) = F(x, t)\frac{du}{dt} + (\Delta_t + \lambda)u(x, t)$$
$$= (F(x, t)N_t^+ + \Delta_t + \lambda)u(x, t).$$

Hence $\frac{dN_t^+}{dt} + (N_t^+)^2 = F(x, t)N_t^+ + (\Delta_t + \lambda)$, so

$$\frac{dN_t^+}{dt} = -(N_t^+)^2 + F(x, t)N_t^+ + (\Delta_t + \lambda).$$

Let

$$\sigma(N_t^+) \sim \alpha_1 + \alpha_0 + \cdots + \alpha_{1-i} + \cdots,$$
$$\sigma(N_t^-) \sim \beta_1 + \beta_0 + \cdots + \beta_{1-i} + \cdots,$$
$$\sigma(\Delta + \lambda) \sim (\sigma_2 + \lambda) + \sigma_1 + \sigma_0.$$

Note that

$$\sigma_2 + \lambda = \left(\sum_{ij=1}^{d-1} g^{ij}\xi_i\xi_j + \lambda\right) Id,$$

$$\sigma((N_t^+)^2) \sim \sum_{k=0}^{\infty} \sum_{\substack{|\omega|+i+j=k \\ i, j \geq 0}} \frac{1}{\omega!} d_\xi^\omega \alpha_{1-i} D_x^\omega \alpha_{1-j},$$

where ω is a multi-index and $D_x = \frac{1}{i}\frac{d}{dx}$.

Since $\frac{dN_t^+}{dt}$, $\frac{dN_t^-}{dt}$ are first order operators, $-\alpha_1^2 + (\sigma_2 + \lambda) = 0$ and $\beta_1^2 - (\sigma_2 + \lambda) = 0$. So

$$\alpha_1 = \beta_1 = \sqrt{\sum_{ij=1}^{d-1} g^{ij}\xi_i\xi_j + \lambda Id} \quad \text{and} \quad \alpha_1 + \beta_1 = 2\sqrt{\sum_{ij=1}^{d-1} g^{ij}\xi_i\xi_j + \lambda Id},$$

which is even with respect to ξ. Note that $\frac{d\alpha_1}{dt} = -(2\alpha_0\alpha_1 + d_\xi\alpha_1 \cdot D_x\alpha_1) + F\alpha_1 + \sigma_1$ and $\frac{d\beta_1}{dt} = (2\beta_0\beta_1 + d_\xi\beta_1 \cdot D_x\beta_1) + F\beta_1 - \sigma_1$. Hence

$$\alpha_0 = \frac{1}{2}\alpha_1^{-1}\left(-\frac{d\alpha_1}{dt} - d_\xi\alpha_1 \cdot D_x\alpha_1 + F\alpha_1 + \sigma_1\right),$$

$$\beta_0 = \frac{1}{2}\beta_1^{-1}\left(\frac{d\beta_1}{dt} - d_\xi\beta_1 \cdot D_x\beta_1 - F\beta_1 + \sigma_1\right).$$

Since $\alpha_1 = \beta_1$, it follows that $\alpha_0 + \beta_0 = \alpha_1^{-1}(d_\xi\alpha_1 \cdot D_x\alpha_1 + \sigma_1)$, which is odd with respect to ξ.

Theorem. *If $\sigma(R(\lambda)) \sim p_1 + p_0 + \cdots + p_{1-j} + \ldots$, then p_{1-k}, which is equal to $-\alpha_{1-k} - \beta_{1-k}$, is even (odd) with respect to ξ when k is even (odd).*

Proof. Note that one has

(*)

$$\begin{cases} \alpha_{1-k} = \frac{1}{2}\alpha_1^{-1}\left\{-\frac{d\alpha_{1-(k-1)}}{dt} - \sum_{\substack{i+j+|\omega|=k \\ 0\le i,\, j\le k-1}} \frac{1}{\omega!} d_\xi^\omega \alpha_{1-i} D_x^\omega \alpha_{1-j} + F\alpha_{1-(k-1)}\right\}, \\ \beta_{1-k} = \frac{1}{2}\beta_1^{-1}\left\{\frac{d\beta_{1-(k-1)}}{dt} - \sum_{\substack{i+j+|\omega|=k \\ 0\le i,\, j\le k-1}} \frac{1}{\omega!} d_\xi^\omega \beta_{1-i} D_x^\omega \beta_{1-j} - F\beta_{1-(k-1)}\right\}. \end{cases}$$

Since $\alpha_1 = \beta_1 = \sqrt{\sum_{ij=1}^{d-1} g^{ij}\xi_i\xi_j + \lambda Id}$, we can use (*) for each α_{1-i}, β_{1-j} to express α_{1-k} and β_{1-k} in terms of α_1, σ_1, and σ_0. In fact,

$$\begin{aligned} \alpha_{1-k} = {} & \sum_r (-1)^r \frac{1}{2}\alpha_1^{-1}\frac{d}{dt}\left\{\frac{1}{2}\alpha_1^{-1}\left\{\frac{d}{dt}\cdots\frac{1}{2}\alpha_1^{-1}\left(\frac{d}{dt}q_r^{k-r}\right)\cdots\right\}\right\} \\ & + \sum_s (-1)^s \frac{1}{2}\alpha_1^{-1} F\left\{\frac{1}{2}\alpha_1^{-1}\left\{F\cdots\frac{1}{2}\alpha_1^{-1}(F\tilde{q}_s^{k-s})\cdots\right\}\right\} + P_k \end{aligned}$$

and

$$\begin{aligned} \beta_{1-k} = {} & \sum_r (-1)^r \frac{1}{2}\alpha_1^{-1}\frac{d}{dt}\left\{\frac{1}{2}\alpha_1^{-1}\left\{\frac{d}{dt}\cdots\frac{1}{2}\alpha_1^{-1}\left(\frac{d}{dt}q_r^{k-r}\right)\cdots\right\}\right\} \\ & + \sum_s (-1)^s \frac{1}{2}\alpha_1^{-1} F\left\{\frac{1}{2}\alpha_1^{-1}\left\{F\cdots\frac{1}{2}\alpha_1^{-1}(F\tilde{q}_s^{k-s})\cdots\right\}\right\} + P_k, \end{aligned}$$

where $\frac{d}{dt}$ appears r times and F appears s times, respectively, and q_r^{k-r}, $\tilde{q}_s^{k-s}$, P_k are functions consisting of some jets of α_1, α_1^{-1}, σ_1, and σ_0 satisfying

$$q_r^{k-r}(x, -\xi) = (-1)^{k-r} q_r^{k-r}(x, \xi),$$

$$\tilde{q}_s^{k-s}(x, -\xi) = (-1)^{k-s} \tilde{q}_s^{k-s}(x, \xi),$$

$$P_k(x, -\xi) = (-1)^k P_k(x, \xi).$$

Hence

$$\begin{aligned} -p_{1-k} &= \alpha_{1-k} + \beta_{1-k} \\ &= 2\sum_{r\,:\ \text{even}} \frac{1}{2}\alpha^{-1}\frac{d}{dt}\left\{\frac{1}{2}\alpha_1^{-1}\left\{\frac{d}{dt}\cdots\frac{1}{2}\alpha_1^{-1}\left(\frac{d}{dt}q_r^{k-r}\right)\cdots\right\}\right\} \\ &\quad + 2\sum_{s\,:\ \text{even}} \frac{1}{2}\alpha_1^{-1} F\left\{\frac{1}{2}\alpha_1^{-1}\left\{F\cdots\frac{1}{2}\alpha_1^{-1}(F\tilde{q}_s^{k-s})\cdots\right\}\right\} + 2P_k, \end{aligned}$$

and so p_{1-k} is even if k is even, and p_{1-k} is odd if k is odd, with respect to ξ.

III. The proof of Theorem B

Lemma 2. *$R(\varepsilon)^{-1} = J \circ (\Delta+\varepsilon)^{-1} \circ (\cdot \otimes \delta_\Gamma)$, where J is the restriction map to Γ and δ_Γ is the Dirac δ-function along Γ.*

Proof. For $\varphi \in C^\infty(\Gamma, E|_\Gamma)$ choose u such that $(\Delta+\varepsilon)u = 0$ in $M - \Gamma$ and $u|_\Gamma = \varphi$. Then

$$\frac{du}{dt} = \begin{cases} \nabla_{\nu_t} u = N_t^+(u(x,t)) & \text{for } t > 0, \\ -\nabla_{-\nu_t} u = -N_t^-(u(x,t)) & \text{for } t < 0. \end{cases}$$

Now $R(\varepsilon)\varphi = -N_0^+(\varphi) - N_0^-(\varphi)$. So

$$\frac{du}{dt} = \begin{cases} -R(\varepsilon)\varphi + N_t^+(u(x,t)) + R(\varepsilon)\varphi, & t \geq 0, \\ -N_t^-(u(x,t)), & t < 0. \end{cases}$$

Let

$$v(x,t) = \begin{cases} N_t^+(u(x,t)) + R(\varepsilon)\varphi, & t \geq 0, \\ -N_t^-(u(x,t)), & t < 0. \end{cases}$$

Then

$$\frac{du}{dt} = -R(\varepsilon)(\varphi) \otimes H(t) + v(x,t).$$

For $t \geq 0$,

$$\begin{aligned} \frac{dv}{dt}(x,t) &= \frac{d}{dt} N_t^+(u(x,t)) = \left\{ \frac{dN_t^+}{dt} + (N_t^+)^2 \right\} u(x,t) \\ &= (F(x,t)N_t^+ + \Delta_t + \varepsilon)u(x,t) \end{aligned}$$

by Lemma 1. In the same way for $t < 0$, $\frac{dv}{dt} = (-F(x,t)N_t^- + \Delta_t + \varepsilon)u(x,t)$. Hence

$$\begin{aligned} \frac{d^2u}{dt^2} &= -R(\varphi) \otimes \delta_\Gamma + \frac{dv}{dt}(x,t) \\ &= -R(\varphi) \otimes \delta_\Gamma + (F(x,t)N_t^+ + \Delta_t + \varepsilon)u(x,t), \end{aligned}$$

$$-\frac{d^2u}{dt^2} + (F(x,t)N_t^+ + \Delta_t + \varepsilon)u(x,t) = R(\varphi) \otimes \delta_\Gamma,$$
$$(\Delta+\varepsilon)u = R(\varphi) \otimes \delta_\Gamma.$$

Hence

$$R(\varepsilon)^{-1}(\varphi) = J \circ (\Delta+\varepsilon)^{-1} \circ (\varphi \otimes \delta_\Gamma).$$

Theorem B. $\operatorname{Det}^*(\Delta) = \frac{1}{(\det A)^2} \operatorname{Det}(\Delta, B) \cdot \operatorname{Det}^* R$.

Proof. Let $k = \dim \mathscr{H}_p$. Then

$$\log \operatorname{Det}(\Delta+\varepsilon) = k \log \varepsilon + \log \operatorname{Det}^*(\Delta) + o(\varepsilon). \tag{1}$$

Denote by $\mu_j = \mu_j(\varepsilon)$ $(j \geq 1)$ the eigenvalues of $R(\varepsilon)$ with $0 < \mu_1(\varepsilon) \leq \cdots \leq \mu_k(\varepsilon) < \mu_{k+1}(\varepsilon) \leq \cdots$. It is clear that $\lim_{\varepsilon \to 0} \mu_j(\varepsilon) = 0$ for $1 \leq j \leq k$. Then

$$\log \operatorname{Det} R(\varepsilon) = \log \mu_1(\varepsilon) \cdots \mu_k(\varepsilon) + \log \operatorname{Det}^* R + o(\varepsilon).$$

Now we want to calculate $\mu_1(\varepsilon)\cdots\mu_k(\varepsilon)$. Let $\{\psi_j\}_{j\geq 1}$ be the complete orthonormal system of eigenforms of Δ with eigenvalue λ_j in $L^2(M, E)$. For any $\varphi \in C^\infty(\Gamma, E|_\Gamma)$, $\varphi\otimes\delta_\Gamma \in H^{-1}(M, E)$ and $(\Delta+\varepsilon)^{-1}(\varphi\otimes\delta_\Gamma) \in L^2(M, E)$.

$$\begin{aligned}\langle(\Delta+\varepsilon)^{-1}(\varphi\otimes\delta_\Gamma), \psi_j\rangle &= \langle\varphi\otimes\delta_\Gamma, (\Delta+\varepsilon)^{-1}\psi_j\rangle = \langle\varphi\otimes\delta_\Gamma, \frac{1}{\lambda_j+\varepsilon}\psi_j\rangle \\ &= \frac{1}{\lambda_j+\varepsilon}\int_\Gamma(\varphi, \psi_j)\,d\mu_\Gamma,\end{aligned}$$

where $d\mu_\Gamma$ is a volume element in Γ. Hence

$$(\Delta+\varepsilon)^{-1}(\varphi\otimes\delta_\Gamma) = \sum_{j=1}^{\infty}\frac{1}{\lambda_j+\varepsilon}\int_\Gamma(\varphi, \psi_j)\,d\mu_\Gamma\cdot\psi_j.$$

Let $\psi_1, \dots, \psi_k$ be harmonic forms and $\lambda_1 = \cdots = \lambda_k = 0$. Then

$$R(\varepsilon)^{-1}\varphi = \frac{1}{\varepsilon}\sum_{i=1}^{k}\int_\Gamma(\varphi, \psi_i)\,d\mu_\Gamma\cdot\psi_i|_\Gamma + \sum_{j=k+1}^{\infty}\frac{1}{\lambda_j+\varepsilon}\int_\Gamma(\varphi, \psi_j)\,d\mu_\Gamma\cdot\psi_j|_\Gamma. \tag{2}$$

From (2), one can check that $R(\varepsilon)^{-1}$ is symmetric and positive definite; it follows that $R(\varepsilon)$ is also symmetric and positive definite.

Let $\phi_1(\varepsilon), \dots, \phi_k(\varepsilon)$ be orthonormal eigenforms of $R(\varepsilon)$ corresponding to eigenvalues $\mu_1(\varepsilon), \dots, \mu_k(\varepsilon)$. Then $\phi_j(\varepsilon) \to \phi_j$ as $\varepsilon \to 0$, where ϕ_j is the restriction of a harmonic form to Γ with $\langle\phi_j, \phi_j\rangle_\Gamma = 1$. Let $a_{ij}(\varepsilon) = \langle\psi_i, \phi_j(\varepsilon)\rangle_\Gamma$, $1 \leq i, j \leq k$, and $A(\varepsilon) = (a_{ij}(\varepsilon))$. Now $\psi_i|_\Gamma = a_{ij}(\varepsilon)\phi_j(\varepsilon) + \tilde{\psi}_i(\varepsilon)|_\Gamma$ for some $\tilde{\psi}_i(\varepsilon)|_\Gamma \in (\operatorname{span}\{\phi_1(\varepsilon), \dots, \phi_k(\varepsilon)\})^\perp$. Define

$$I\colon C^\infty(\Gamma, E|_\Gamma) \to C^\infty(\Gamma, E|_\Gamma)$$

by

$$\varphi \mapsto \sum_{j=1}^{k}\int_\Gamma(\varphi, \psi_j)\,d\mu_\Gamma\cdot\psi_j|_\Gamma = \sum_{j=1}^{k}\langle\varphi, \psi_j\rangle_\Gamma\cdot\psi_j|_\Gamma.$$

Then

$$\langle I(\phi_i(\varepsilon)), \phi_j(\varepsilon)\rangle_\Gamma = \sum_{l=1}^{k}a_{li}(\varepsilon)a_{lj}(\varepsilon) = ({}^tAA)_{ij}(\varepsilon).$$

Define

$$G_\varepsilon\colon C^\infty(\Gamma, E|_\Gamma) \to C^\infty(\Gamma, E|_\Gamma)$$

by

$$\varphi \mapsto \sum_{j=k+1}^{\infty}\frac{1}{\lambda_j+\varepsilon}\langle\varphi, \psi_j\rangle_\Gamma\cdot\psi_j|_\Gamma.$$

Then $\|G_\varepsilon\|_{L^2}$ converges to $\frac{1}{\lambda_{k+1}} > 0$ as $\varepsilon \to 0$. Now

$$R(\varepsilon)^{-1}(\varphi) = \frac{1}{\varepsilon}I(\varphi) + G_\varepsilon(\varphi).$$

For $1 \leq j \leq k$,

$$\begin{aligned} \frac{1}{\mu_j(\varepsilon)} &= \langle R(\varepsilon)^{-1}\phi_j(\varepsilon), \phi_j(\varepsilon)\rangle \\ &= \frac{1}{\varepsilon}\langle I(\phi_j(\varepsilon)), \phi_j(\varepsilon)\rangle + \langle G_\varepsilon(\phi_j(\varepsilon)), \phi_j(\varepsilon)\rangle \\ &= \frac{1}{\varepsilon}({}^tAA)_{jj}(\varepsilon) + N_j(\varepsilon), \end{aligned}$$

where $N_j(\varepsilon) = \langle G_\varepsilon(\phi_j(\varepsilon)), \phi_j(\varepsilon)\rangle_\Gamma$ is bounded as $\varepsilon \to 0$. For $i \neq j$, $1 \leq i, j \leq k$,

$$\begin{aligned} 0 &= \langle R(\varepsilon)^{-1}(\phi_i(\varepsilon)), \phi_j(\varepsilon)\rangle \\ &= \frac{1}{\varepsilon}\langle I(\phi_i(\varepsilon)), \phi_j(\varepsilon)\rangle + \langle G_\varepsilon(\phi_i(\varepsilon)), \phi_j(\varepsilon)\rangle \\ &= \frac{1}{\varepsilon}({}^tAA)_{ij}(\varepsilon) + \langle G_\varepsilon(\phi_i(\varepsilon)), \phi_j(\varepsilon)\rangle. \end{aligned}$$

Since $({}^tAA)_{ij}(\varepsilon)$ and $\langle G_\varepsilon(\phi_i(\varepsilon)), \phi_j(\varepsilon)\rangle$ are bounded, $(tAA)_{ij}(\varepsilon) \to 0$ as $\varepsilon \to 0$. So

$$\begin{aligned} \frac{1}{\mu_1(\varepsilon)\cdots\mu_k(\varepsilon)} &= \left(\frac{1}{\varepsilon}({}^tAA)_{11} + N_1(\varepsilon)\right)\cdots\left(\frac{1}{\varepsilon}({}^tAA)_{kk} + N_k(\varepsilon)\right) \\ &= \frac{1}{\varepsilon^k}(\det A)^2\left(\frac{({}^tAA)_{11}({}^tAA)_{22}\cdots({}^tAA)_{kk}}{(\det A)^2} + \varepsilon\cdot\frac{\widetilde{N}(\varepsilon)}{(\det A)^2}\right), \end{aligned}$$

where $\widetilde{N}(\varepsilon)$ is bunded as $\varepsilon \to 0$. Hence

$$\log \operatorname{Det} R(\varepsilon) = k\log\varepsilon - \log(\det A)^2 + \log \operatorname{Det}^* R + o(\varepsilon). \tag{3}$$

If we combine equation (1) and equation (3), we get

$$\log \operatorname{Det}^* \Delta = -\log(\det A)^2 + \log \operatorname{Det}^* R + \log \operatorname{Det}(\Delta, B).$$

Acknowledgements

I thank D. Burghelea for asking me to settle the questions answered by Theorems A and B in this paper.

References

[BFK] D. Burghelea, L. Friedlander, and T. Kappeler, *Mayer-Vietoris type formula for determinants of elliptic differential operators*, J. Funct. Anal. **107** (1992), 34.

Department of Mathematics, Ohio State University, Columbus, Ohio 43210
E-mail address: ywlee@math.ohio-state.edu