# SOME COMPLETE RICCI-FLAT KAHLER METRICS IN $\mathbb{C} P^{\mathbf{2}}$ 

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#### Abstract

In this article, we give an explicit construction of complete Ricci-flat Kahler metrics in $\mathbb{C} P^{\mathbf{2}} \backslash\{$ three complex lines in general position $\}$.


## 1. Introduction

In [3], S.-T. Yau proved the existence of Ricci-flat metrics on compact Kahler manifolds, it then seems natural to consider the same problem on noncompact Kahler manifolds. For general existence theory, we refer the interested readers to [1] and [2]. In this article, we can explicitly construct some solutions of complete Ricci-flat Kahler metrics in $\mathbb{C} P^{2} \backslash\{$ three complex lines in general position $\}$.

## 2. Affine consideration

Given a Kahler manifold ( $M^{n}, \omega$ ) , it is well known that its associated Riccitensor is $(-) \partial_{i} \bar{\partial}_{j} \log \operatorname{det}\left(g_{i j}\right)$, where $\omega=-\sqrt{-1} g_{i j} d z^{i} \wedge d \bar{z}^{j}$ is the Kahler form of the manifold. Also, notice that the $\operatorname{det}\left(g_{i j}\right)$ is the associated volume form, and the condition of Ricci-flatness is equivalent to the equation $\partial \bar{\partial} \log \operatorname{det}\left(g_{i j}\right)=0$.

Specifically, in this article, we are interested in constructing complete Ricciflat Kahler metrics in $\mathbb{C} P^{2} \backslash\{$ three complex lines in general position $\}$. Using the automorphism group of $\mathbb{C} P^{2}$, which is $\operatorname{PGL}(3, \mathbb{C})$, without loss of generality, we may assume that our noncompact Kahler manifold is $\mathbb{C} P^{2} \backslash\{x=0, y=$ $0, x+y+1=0\}$, where $(x, y) \in \mathbb{C}^{2}$ denotes the complex coordinate system in $\mathbb{C}^{2}$. And to construct complete Ricci-flat Kahler metrics is equivalent to solving for the associated Kahler potential $W$, of the following form:

$$
\begin{equation*}
\operatorname{det}\left(\partial_{i} \bar{\partial}_{j} W\right)=|x y(x+y+1)|^{-2} \quad \text { in } \mathbb{C}^{2} \backslash\{x y(x+y+1)=0\} \tag{2.1}
\end{equation*}
$$

If so, then the Kahler metric is given by $d s^{2}=\left(\partial_{i} \bar{\partial}_{j} W\right) d z^{i} \otimes d \bar{z}^{j}$.
In constructing explicit solutions of (2.1), we restrict ourselves, in this article, to special Kahler potentials $W$, of the following form:
(2.2) $W=\phi \circ f+\psi \circ g$ where $\phi, \psi: \mathbb{C} \rightarrow \mathbb{R}$ are smooth functions and $f, g: \mathbb{C}^{2} \rightarrow \mathbb{C}$ are meromorphic functions.

Simple computations imply that equation (2.1) is equivalent to the following: (2.3)
$(\partial \bar{\partial} \phi)(\partial \bar{\partial} \psi)|J(f, g)|^{2}=|x y(x+y+1)|^{-2}$ in $\mathbb{C}^{2} \backslash\{x y(x+y+1)=0\}$,
where $J(f, g)$ is the Jacobian of $(f, g)$.
So, our goal is to solve for $\phi, \psi, f$ and $g$ such that $d s^{2}=\left(\partial_{i} \bar{\partial}_{j} W\right) d z^{i} \otimes d \bar{z}^{j}$ gives us complete metrics in $\mathbb{C} P^{2} \backslash\{x y(x+y+1)=0\}$, which, more precisely, is $\mathbb{C} P^{2} \backslash\{x y(x+y+z)=0\}$ if we use the homogeneous coordinate system $(x: y: z) \in \mathbb{C} P^{2}$ with the line at infinity defined as $\{z=0\}$.

First, we have the following lemma:
Lemma 2.1. The only possible solutions $f, g$ of equation (2.3) are of the following form:

$$
\begin{align*}
& f=x^{a_{1}} y^{b_{1}}(x+y+1)^{c_{1}}, \text { for some integers } a_{1}, b_{1}, \text { and } c_{1}  \tag{2.4}\\
& g=x^{a_{2}} y^{b_{2}}(x+y+1)^{c_{2}}, \text { for some integers } a_{2}, b_{2}, \text { and } c_{2} \tag{2.5}
\end{align*}
$$

Proof. First of all, we note that equation (2.3) is equivalent to

$$
\begin{equation*}
(\partial \bar{\partial} \phi \circ f)(\partial \bar{\partial} \psi \circ g)=|J(f, g) x y(x+y+1)|^{-2} \tag{2.6}
\end{equation*}
$$

Secondly, note that the left-hand side of equation (2.6) is of the product form $\tilde{\phi}(f) \cdot \tilde{\psi}(g)$; therefore, the right-hand side must be of the same form. That is, for some funtions $\widetilde{\tilde{\phi}}, \widetilde{\widetilde{\psi}}$, we have

$$
\begin{equation*}
J(f, g) x y(x+y+1)=\widetilde{\widetilde{\phi}}(f) \cdot \tilde{\widetilde{\psi}}(g) \tag{2.7}
\end{equation*}
$$

A careful investigation of the level curves of $\tilde{\tilde{\phi}}$ and $\tilde{\tilde{\psi}}$ leads to the conclusion that $\tilde{\widetilde{\phi}}(f)=f^{\alpha}$ and $\widetilde{\widetilde{\psi}}(g)=g^{\beta}$, for some constants $\alpha$ and $\beta$. That is, we have

$$
\begin{gather*}
J(f, g) x y(x+y+1)=f^{\alpha} g^{\beta},  \tag{2.8}\\
\partial \bar{\partial} \phi=|f|^{-2 \alpha} \text { and } \partial \bar{\partial} \psi=|g|^{-2 \beta},  \tag{2.9}\\
d s^{2}=\frac{f_{x} \bar{f}_{x} \cdot|f|^{-2 \alpha}+g_{x} \bar{g}_{x} \cdot|g|^{-2 \beta}}{f_{y} \bar{f}_{x} \cdot|f|^{-2 \alpha}+g_{y} \bar{g}_{x} \cdot|g|^{-2 \beta}} \frac{f_{x} \bar{y}_{y} \cdot|f|^{-2 \alpha}+g_{x} \bar{g}_{y} \cdot|g|^{-2 \beta}}{f_{y} \bar{f}_{y} \cdot|f|^{-2 \alpha}+g_{y} \bar{g}_{y} \cdot|g|^{-2 \beta}} . \tag{2.10}
\end{gather*}
$$

From the expression of $d s^{2}$ above, equation (2.8) and the assumption that $f, g$ are meromorphic in $\mathbb{C}^{2}$, the desired conclusion follows easily.

Actually, we know more about $f$ and $g$, as shown by the following:
Lemma 2.2. The only possible constants for equation (2.8) to be true are $\alpha=$ $1, \beta=1$, and $a_{i}, b_{i}, c_{i}$ integers, $i=1,2$, with $a_{1}+b_{1}+c_{1}=0$ and $a_{2}+$ $b_{2}+c_{2}=0$.
Proof. Let $f=x^{a_{1}} y^{b_{1}}(x+y+1)^{c_{1}}$ and $g=x^{a_{2}} y^{b_{2}}(x+y+1)^{c_{2}}$. Then

$$
\begin{aligned}
& J(f, g) x y(x+y+1) \\
& \quad=x^{a_{1}+a_{2}} y^{b_{1}+b_{2}}(x+y+1)^{c_{1}+c_{2}}\left(\left(a_{1} b_{2}-a_{2} b_{1}+c_{1} b_{2}-c_{2} b_{1}\right) x\right. \\
& \left.\quad+\left(a_{1} b_{2}-a_{2} b_{1}+a_{1} c_{2}-a_{2} c_{1}\right) y+\left(a_{1} b_{2}-a_{2} b_{1}\right)\right) .
\end{aligned}
$$

So, if $f, g$ satisfy (2.8), we will have the following possibilities:
(a)

$$
\begin{aligned}
& a_{1} b_{2}-a_{2} b_{1}+c_{1} b_{2}-c_{2} b_{1}=0=a_{1} b_{2}-a_{2} b_{1}+a_{1} c_{2}-a_{2} c_{1} \\
& a_{1} b_{2}-a_{2} b_{1} \neq 0
\end{aligned}
$$

and

$$
\begin{aligned}
& a_{1}+a_{2}=\alpha a_{1}+\beta a_{2} \\
& b_{1}+b_{2}=\alpha b_{1}+\beta b_{2} \\
& c_{1}+c_{2}=\alpha c_{1}+\beta c_{2}
\end{aligned}
$$

or
(b)

$$
\begin{aligned}
& a_{1} b_{2}-a_{2} b_{1}+c_{1} b_{2}-c_{2} b_{1}=0=a_{1} b_{2}-a_{2} b_{1} \\
& a_{1} b_{2}-a_{2} b_{1}+a_{1} c_{2}-a_{2} c_{1} \neq 0
\end{aligned}
$$

and

$$
\begin{aligned}
& a_{1}+a_{2}=\alpha a_{1}+\beta a_{2} \\
& b_{1}+b_{2}+1=\alpha b_{1}+\beta b_{2} \\
& c_{1}+c_{2}=\alpha c_{1}+\beta c_{2}
\end{aligned}
$$

or
(c)

$$
\begin{aligned}
& a_{1} b_{2}-a_{2} b_{1}+a_{1} c_{2}-a_{2} c_{1}=0=a_{1} b_{2}-a_{2} b_{1} \\
& a_{1} b_{2}-a_{2} c_{1}+c_{1} b_{2}-c_{2} b_{1} \neq 0
\end{aligned}
$$

and

$$
\begin{aligned}
& a_{1}+a_{2}+1=\alpha a_{1}+\beta a_{2} \\
& b_{1}+b_{2}=\alpha b_{1}+\beta b_{2} \\
& c_{1}+c_{2}=\alpha c_{1}+\beta c_{2}
\end{aligned}
$$

or
(d) $a_{1} b_{2}-a_{2} b_{1}+c_{1} b_{2}-c_{2} b_{1}=a_{1} b_{2}-a_{2} b_{1}+a_{1} c_{2}-a_{2} c_{1}=a_{1} b_{2}-a_{2} b_{1} \neq 0$
and

$$
\begin{aligned}
& a_{1}+a_{2}=\alpha a_{1}+\beta a_{2} \\
& b_{1}+b_{2}=\alpha b_{1}+\beta b_{2} \\
& c_{1}+c_{2}+1=\alpha c_{1}+\beta c_{2}
\end{aligned}
$$

Finally, after some simple computations, it is easy to see that only case (a) is possible and for this case, we have $\alpha=1, \beta=1, a_{1}+b_{1}+c_{1}=0$ and $a_{2}+b_{2}+c_{2}=0$.

In summary, the special Kahler potential of the form (2.2) with $f=$ $x^{a_{1}} y^{b_{1}}(x+y+1)^{c_{1}}, g=x^{a_{2}} y^{b_{2}}(x+y+1)^{c_{2}}, a_{1}+b_{1}+c_{1}=0, a_{2}+b_{2}+c_{2}=0$, satisfies (2.1) and it gives a metric

$$
\begin{equation*}
d s^{2}=\frac{f_{x} \bar{f}_{x} \cdot|f|^{-2}+g_{x} \bar{g}_{x} \cdot|g|^{-2}}{f_{y} \bar{f}_{x} \cdot|f|^{-2}+g_{y} \bar{g}_{x} \cdot|g|^{-2}} \frac{f_{x} \bar{f}_{y} \cdot|f|^{-2}+g_{x} \bar{g}_{y} \cdot|g|^{-2}}{f_{y} \bar{f}_{y} \cdot|f|^{-2}+g_{y} \bar{g}_{y} \cdot|g|^{-2}} . \tag{2.11}
\end{equation*}
$$

We need further verify that (1) $d s^{2}$ can be compactified to give a smooth metric at the line of infinity, $\mathbb{C} P^{1} ;(2) d s^{2}$ is complete in neighborhoods of the deleted set, $\{x y(x+y+1)=0\}$. We will do this in the next section.

## 3. Projective consideration

First, we do some changes of variables.
In a neighborhood of $(1: 0: 0) \in \mathbb{C} P^{2}$, we use the new coordinate system $(u, v)$, where

$$
\begin{equation*}
u=\frac{1}{x} \quad \text { and } \quad v=\frac{y}{x} \tag{3.1}
\end{equation*}
$$

or equivalently

$$
\begin{equation*}
x=\frac{1}{u} \quad \text { and } \quad y=\frac{v}{u} . \tag{3.2}
\end{equation*}
$$

Expressing the metric in terms of the new coordinate system $(u, v)$, we have

$$
\begin{align*}
& d s^{2}=\frac{\frac{1}{|u|^{4}}\left(\left|f_{x}+v f_{y}\right|^{2} \cdot|f|^{-2}+\left|g_{x}+v f_{y}\right| \cdot|g|^{-2}\right)}{\frac{\frac{-u}{\mid u 4^{4}}\left(\left(f_{x}+v f_{y}\right) f_{y} \cdot|f|^{-2}\right)}{+\frac{-u}{|u|^{4}}\left(\left(g_{x}+0 g_{y}\right) g_{y} \cdot|g|^{-2}\right)}} \\
& \frac{\frac{-\frac{q}{\mid u 4^{4}}\left(\left(f_{x}+v f_{y}\right) f_{y} \cdot|f|^{-2}\right)}{+\frac{-u^{4}}{\left.\left.\left|u^{4}\left(\left(g_{x}+v g_{y}\right)\right)_{y} \cdot\right| g\right|^{-2}\right)}}}{\frac{1}{|u|^{2}}\left(f_{y} \cdot \bar{f}_{y} \cdot|f|^{-2}+g_{y} \bar{g}_{y} \cdot|g|^{-2}\right)} . \tag{3.3}
\end{align*}
$$

Moreover, it is easy to check that $d s^{2}$ is smooth across the line of infinity. Similarly, $d s^{2}$ is smooth across the line of infinity in neighborhoods of $0: 1$ : 0 ) and ( $1:-1: 0$ ). So these show that $d s^{2}$ is a smooth metric across the line at infinity.

Next, we will show that $d s^{2}$ is a complete metrc in a neighborhood of the set $\{x y(x+y+1)=0\}$. Without loss of generality, we need only study the behavior of $d s^{2}$ in a neighborhood of $\{x=0\}$. But in a small neighborhood of $\left(0, y_{0}\right) \in \mathbb{C}^{2}$, we have

$$
\begin{align*}
d s^{2}= & \frac{\left|\left(a_{1}+c_{1}\right) x+a_{1} y+a_{1}\right|^{2}+\left|\left(a_{2}+c_{2}\right) x+a_{2} y+a_{2}\right|^{2}}{|x(x+y+1)|^{2}} d x d \bar{x} \\
& +\left[\frac{\left(\left(a_{1}+c_{1}\right) x+a_{1} y+a_{1}\right)\left(\bar{c}_{1} \bar{x}+\left(\bar{b}_{1}+\bar{c}_{1}\right) \bar{y}+\bar{b}_{1}\right)}{x \bar{y}|(x+y+1)|^{2}}\right. \\
& \left.+\frac{\left(\left(a_{2}+c_{2}\right) x+a_{2} y+a_{2}\right)\left(\bar{c}_{2} \bar{x}+\left(\bar{b}_{2}+\bar{c}_{2}\right) \bar{y}+\bar{b}_{2}\right)}{x \bar{y}|(x+y+1)|^{2}}\right] d x d \bar{y}  \tag{3.4}\\
& +\left[\frac{\left(\left(\bar{a}_{1}+\bar{c}_{1}\right) \bar{x}+\bar{a}_{1} \bar{y}+\bar{a}_{1}\right)\left(c_{1} x+\left(b_{1}+c_{1}\right) y+b_{1}\right)}{x \bar{y}|(x+y+1)|^{2}}\right. \\
& \left.+\frac{\left(\left(\bar{a}_{2}+\bar{c}_{2}\right) \bar{x}+\bar{a}_{2} \bar{y}+\bar{a}_{2}\right)\left(c_{2} x+\left(b_{2}+c_{2}\right) y+b_{2}\right)}{x \bar{y}|(x+y+1)|^{2}}\right] d \bar{x} d y \\
& +\frac{\left.\mid c_{1} x+c b_{1}+c_{1}\right) y+\left.b_{1}\right|^{2}+\left|c_{2} x+\left(b_{2}+c_{2}\right) y+b_{2}\right|^{2}}{|y(x+y+1)|^{2}} d y d \bar{y}
\end{align*}
$$

$$
\begin{aligned}
\approx & \frac{\left|a_{1} y_{0}+a_{1}\right|^{2}+\left|a_{2} y_{0}+a_{2}\right|^{2}}{\left|x\left(y_{0}+1\right)\right|^{2}} d x d \bar{x} \\
& +\frac{\left(a_{1} y_{0}+a_{1}\right)\left(\left(\bar{b}_{1}+\bar{c}_{1}\right) \bar{y}_{0}+\bar{b}_{1}\right)+\left(a_{2} y_{0}+a_{2}\right)\left(\left(\bar{b}_{2}+\bar{c}_{2}\right) \bar{y}_{0}+\bar{b}_{2}\right)}{x \bar{y}_{0}\left|y_{0}+1\right|^{2}} d x d \bar{y} \\
& +\frac{\left(\bar{a}_{1} \bar{y}_{0}+\bar{a}_{1}\right)\left(\left(b_{1}+c_{1}\right) y_{0}+b_{1}\right)+\left(\bar{a}_{2} \bar{y}_{0}+\bar{a}_{2}\right)\left(\left(b_{2}+c_{2}\right) y_{0}+b_{2}\right)}{\bar{x} y_{0}\left|y_{0}+1\right|^{2}} d \bar{x} d y \\
& +\frac{\left|\left(b_{1}+c_{1}\right) y_{0}+b_{1}\right|^{2}+\left|\left(b_{2}+c_{2}\right) y_{0}+b_{2}\right|^{2}}{\left|y_{0}\left(y_{0}+1\right)\right|^{2}} d y d \bar{y} .
\end{aligned}
$$

This asymptotic behavior of $d s^{2}$ readily shows that $d s^{2}$ is complete in this neighborhood. In conclusion, we have the following
Theorem 3.1. In $\mathbb{C} P^{2}-\{x y(x+y+z)=0\}$, we have the following complete Ricci-flat metrics:

$$
d s^{2}=\frac{f_{x} \bar{f}_{x} \cdot|f|^{-2}+g_{x} \bar{g}_{x} \cdot|g|^{-2}}{f_{y} \bar{f}_{x} \cdot|f|^{-2}+g_{y} \bar{g}_{x} \cdot|g|^{-2}} \frac{f_{x} \bar{f}_{y} \cdot|f|^{-2}+g_{x} \bar{g}_{y} \cdot|g|^{-2}}{f_{y} \bar{f}_{y} \cdot|f|^{-2}+g_{y} \bar{g}_{y} \cdot|g|^{-2}}
$$

where $f=x^{a_{1}} y^{b_{1}}(x+y+1)^{c_{1}}$ and $g=x^{a_{2}} y^{b_{2}}(x+y+1)^{c_{2}}$ with $a_{i}, b_{i}, c_{i} \in \mathbb{Z}$ and $a_{1}+b_{1}+c_{1}=0, a_{2}+b_{2}+c_{2}=0$.
Remark. Since, for flat metrics on $C P^{2}$ with three generic lines deleted, the associated automorphic group is the free abelian of two generators, some of our constructed metrics in the main theorem definitely would be nonflat, by naive counting. More generalizations along this direction will appear in a forthcoming paper.

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