EXISTENCE AND NONEXISTENCE OF POSITIVE EIGENFUNCTIONS FOR THE p -LAPLACIAN

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ABSTRACT. We study the relation between (i) the principal eigencurve, i.e., the graph of μ_1 satisfying

$$(*) \quad -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + (q(x) - \lambda w(x))|u|^{p-2}u = \mu_1(\lambda)|u|^{p-2}u$$

on a smooth bounded domain Ω in $\mathbf{R}^N$ with $p > 1$, and (ii) existence and nonexistence of positive solutions of $(*)$ with $\mu_1(\lambda) = 0$. Eigencurve arguments are used extensively.

1. INTRODUCTION

We study the partial differential eigenvalue problem

$$(1.1) \quad -\Delta_p u + q[u]_p = \lambda w[u]_p,$$

on a bounded smooth domain $\Omega \subset \mathbf{R}^N$, where $p > 1$, $[u]_p = |u|^{p-2}u$, and $\Delta_p u = \nabla[\nabla u]_p$ is the homogeneous operator called the p -Laplacian. On the boundary Π of Ω we impose (possibly) mixed boundary conditions:

$$(1.2) \quad u = 0 \text{ on } \Pi_1, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Pi_2, \quad \Pi_1 \cup \Pi_2 = \Pi, \quad \Pi_1 \cap \Pi_2 = \emptyset,$$

where either Π_1 or Π_2 could be empty. The real valued functions q and w are assumed to be L^∞ on Ω , and w is not identically zero, but could be indefinite.

In the case $p = 2$, Δ_p is the usual Laplacian, and there is a number of existence and nonexistence results for principal eigenvalues of (1.1), with corresponding eigenfunctions of one sign. For example we refer to [AM] for a treatment of the "indefinite weight" case where w takes both signs and does not vanish on a positive measure set. In this case there are two (linearly independent) eigenfunctions of one sign if and only if

$$(1.3) \quad \text{for some real } \lambda_0, \quad -\Delta + q - \lambda_0 w \text{ is positive definite,}$$

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and it then turns out that for any eigenpair (λ, u) of (1.1),

$$(1.4) \quad (\lambda - \lambda_0) \int_{\Omega} w|u|^p > 0.$$

(In [AM], the statement is slightly different, and in particular they conclude an inequality similar to (1.4) for real eigenvalues λ , as well as

$$(1.5) \quad \int_{\Omega} w|u|^p \neq 0$$

for all λ . Any eigenfunctions u violating (1.5) are called “ghost states” in [AM] by analogy with the case $\lambda \notin \mathbf{R}$. On the other hand, there may also be real eigenpairs (λ, u) generating such “ghost states”—see Remark 4.3 for further discussion.) Eigencurves based on two-parameter embeddings have also proved quite useful for (1.1) in the case of indefinite weight and [BB2], [BBHP] contain several consequences for (1.1), in cases where $w^{-1}(\sigma)$ ($\sigma = \text{ess sup } w$) is suitably restricted.

When $p \neq 2$, Δ_p arises in a variety of applications, for which we refer to [BH1–3] and the references therein. This paper is motivated by a question raised to us by Dr. Allegretto: whether similar results for “ghost states” still exist for the p -Laplacian with $p \neq 2$. We remark that the method used in [AM] relies critically on a Picone type identity, which is not available when $p \neq 2$. On the other hand we find that simple properties of the principal eigencurve for a related two-parameter eigenvalue problem can still be used to study positive eigenfunctions for (1.1). This enables us not only to answer the “ghost state” question, but also to extend some other results about indefinite eigenvalue problems from $p = 2$ to $p \neq 2$. Another advantage of this method is that three (Dirichlet, Neumann and mixed) boundary conditions are covered by a unified approach.

We shall discuss all three possibilities for w —definite, semidefinite and indefinite. In particular, we shall show that if w is indefinite, then the existence of two independent eigenfunctions of one sign is equivalent to the obvious analogue of (1.3) and implies (1.4) for all eigenpairs (λ, u) . Moreover it is easily seen that (1.4) for an eigenpair (λ, u) implies $\lambda \in \mathbf{R}$ and (1.5), but there are examples where (1.4) holds for all real (λ, u) without (1.5) and where (1.5) holds for all eigenpairs (λ, u) without (1.4) [B].

2. PRELIMINARIES

The space in which problem (1.1) is studied is chosen to be

$$X = \{u \in W^{1,p}(\Omega) : u = 0 \text{ on } \Pi_1\}.$$

A (complex valued) function $u \in X$ is a solution of (1.1) subject to boundary conditions (1.2) if for some $\lambda \in \mathbf{C}$ and for any $\varphi \in C_0^\infty(\Omega)$,

$$\int_{\Omega} [\nabla u]_p \nabla \varphi + \int_{\Omega} q[u]_p \varphi = \int_{\Omega} \lambda w[u]_p \varphi.$$

Existence of (variational) eigenvalues and eigenfunctions of (1.1) will be obtained via Ljusternik-Schnirelman theory (cf. [BH1–2]).

Let

$$I(\lambda, u) := \int_{\Omega} |\nabla u|^p + \int_{\Omega} (q - \lambda w)|u|^p$$

and

$$U := \left\{ u \in X : \int_{\Omega} |u|^p = 1 \right\}.$$

First we have the following theorem, which is contained in [BH1, Lemmas 2.1–2.4].

Theorem 2.1. (a) For any $\lambda \in \mathbf{R}$, there exists a unique $\mu_1(\lambda) = \min_{u \in U} I(\lambda, u)$, and $\mu_1(\lambda)$ is attained by a unique positive $u (= u_1(\lambda))$ say) satisfying

$$-\Delta_p u_1(\lambda) + (q - \lambda w)[u_1(\lambda)]_p = \mu_1(\lambda)[u_1(\lambda)]_p \quad \text{in } \Omega.$$

(b) μ_1 is concave and differentiable on $\mathbf{R}$, and $\mu'_1(\lambda) = - \int_{\Omega} w |u_1|^p$.

We remark that the proof of part (a) follows from a variational principle and the maximum principle (cf. [OT]), while part (b) results from the definition of μ and direct calculations.

Theorem 2.2. Let $\sigma = \text{ess sup } w$. Then:

(a) $\mu_1(\lambda) + \sigma\lambda$ is strictly increasing for $\lambda > 0$ if $\int_{\Omega} (w - \sigma) < 0$, and $\mu_1(\lambda) + \sigma\lambda = \mu_1(0)$ if $w = \sigma$ a.e. on Ω .

(b) If $w = \sigma$ a.e. on some nonempty open set $\Omega^+ \subset \Omega$, then

$$(2.1) \quad \mu^+ = \inf \{ I(0, u) : u \in U, \text{supp } u \subset \Omega^+ \}$$

exists, and $\mu_1(\lambda) + \sigma\lambda \leq \mu^+$ for all real λ .

Proof. (a) Differentiating $\mu_1(\lambda) + \sigma\lambda$ yields the result.

(b) Since $\mu_1(\lambda) + \sigma\lambda$ is increasing, suffice it to consider only $\lambda > 0$. Writing $\mu_1^+(\lambda)$ for the eigenvalue $\mu_1(\lambda)$ in Theorem 2.1, but with Ω replaced by Ω^+ , we see that $\mu^+ = \mu_1^+(0)$. Moreover, $\mu_1(\lambda) \leq \mu_1^+(\lambda)$ since the former minimum is over all $u \in U$, so

$$\mu_1(\lambda) + \sigma\lambda \leq \mu_1^+(\lambda) + \sigma\lambda = \mu^+$$

by (a) (again with Ω replaced by Ω^+). $\square$

Remark 2.1. If w does not reach its supremum on some open set, a weak version of Theorem 2.2 (b) still holds. Indeed, assume for some $\varepsilon > 0$, $w > \sigma - \varepsilon$ on some open $\Omega^+ \subset \Omega$ (which is true, for example, if w is piecewise continuous on Ω), and let μ^+ be given by (2.1) with w in the functional I replaced by σ . Then the proof of Theorem 2.2 (b) yields $\mu_1(\lambda) + \sigma\lambda \leq \mu^+ + \lambda\varepsilon$.

3. RIGHT (SEMI) DEFINITE CASES

We assume in this section that $w \leq 0$ a.e. in Ω so $\sigma \leq 0$.

Theorem 3.1. If $w < 0$ a.e. in Ω , then (i) there exist unique $\lambda = \lambda_1$ and positive $u = u_1 \in U$ satisfying

$$(3.1) \quad -\Delta_p u + (q - \lambda w)[u]_p = 0,$$

and (ii) for all other eigenpairs (λ, u) , λ is real, $\lambda < \lambda_1$ and (1.4) holds for all $\lambda_0 > \lambda_1$.

Proof. If $\sigma < 0$, then by Theorem 2.1 (b) $\mu'_1(\lambda) \geq -\sigma$ for all λ so there exists a unique λ_1 so that $\mu_1(\lambda_1) = 0$. We write the corresponding eigenfunction as $u_1 = u_1(\lambda_1)$. If (λ, u) forms another eigenpair, then

$$(3.2) \quad \int_{\Omega} w |u|^p < 0,$$

so $\lambda = I(0, u) / \int w|u|^p$ exists real and moreover $\lambda > \lambda_1$ would imply $\mu_1(\lambda) > 0$ whereas (3.1) implies $I(\lambda, u) = 0$, contradiction. Thus $\lambda < \lambda_1$ and $\lambda_1 < \lambda_0$, which together with (3.2) gives (1.4).

If $\sigma = 0$, μ_1 still increases by Theorem 2.1 (b). We first claim there exists λ_1 such that $\mu_1(\lambda_1) = 0$. If not, then positivity of $\mu'_1(\lambda)$ (by Theorem 2.1 (b)) and concavity of $\mu_1(\lambda)$ imply that, for each $n > 1$, $I(n, u_n) < 0$ for some $u_n \in U$. Since $w < 0$, we obtain $I(0, u_n) < 0$ so $\int_{\Omega} |\nabla u_n|^p$ are bounded. Thus u_n converge (subsequentially) weakly in X , and hence strongly in L^p , to some u in U . We now let $n \rightarrow +\infty$ in the inequality $n^{-1}I(0, u_n) < 0$ to give $\int_{\Omega} w|u|^p \geq 0$, contradiction.

It follows that such λ_1 exists. Uniqueness comes from monotonicity of $\mu_1(\lambda)$ and the rest of the proof follows the case $\sigma < 0$. $\square$

Theorem 3.1 shows that the graph of μ_1 has no horizontal asymptote if $w < 0$ a.e. Next we consider a situation where an asymptote does exist.

Theorem 3.2. *Assume that $\Omega^* := w^{-1}(0)$ is a nonempty open set. Then $\mu_1(\lambda) \rightarrow \mu^*$ as $\lambda \rightarrow +\infty$, where*

$$\mu^* := \inf\{I(0, u) : u \in U, \text{supp } u \subset \Omega^*\}.$$

Proof. Theorem 2.2(b) implies that $\mu_1(\lambda) \leq \mu^*$ since $\sigma = 0$ here, so $\{u_1(\lambda) : \lambda > 0\}$ is bounded in X . It then follows from (the proof of) Theorem 3.4 of [BH1] that there exists a $u_{\infty} \in X$ such that (i) $u_1(\lambda_n) \rightarrow u_{\infty}$ weakly in X and strongly in L^p (for a sequence $\lambda_n \rightarrow +\infty$) and (ii) $w[u_{\infty}]_p = 0$ a.e. in Ω , i.e., $\text{supp}(u_{\infty}) \subset \Omega^+$. Thus since $\mu_1(\lambda)$ is nondecreasing in λ by Theorem 2.1(b), we have

$$\begin{aligned} \lim_{\lambda \rightarrow +\infty} \mu_1(\lambda) &= \lim_{\lambda_n \rightarrow \infty} I(\lambda_n, u_1(\lambda_n)) \geq \liminf_{\lambda_n \rightarrow \infty} \left(\int |\nabla u_1(\lambda_n)|^p + \int q|u_1(\lambda_n)|^p \right) \\ &\geq I(0, u_{\infty}) \quad \text{by (i) (cf. [K, p. 137])} \\ &\geq \mu^* \quad \text{by (ii).} \quad \square \end{aligned}$$

We note that these asymptotic results extend the work of [A] and [BBHP] in certain respects to $p \neq 2$. The following corollary is a partial converse of Theorem 3.4 of [BH1].

Corollary 3.3. *$\{u_1(\lambda) : \lambda > 0\}$ is bounded in X if $w^{-1}(\sigma)$ contains a nonempty open set.*

Corollary 3.4. *If $\mu^* > 0$, then conclusion (i) of Theorem 3.1 holds.*

Remark 3.5. To obtain conclusion (ii) of Theorem 3.1, it seems to be necessary to assume a form of unique continuation. Specifically, suppose we assume that if a solution u of (3.1) vanishes on

$$(3.3) \quad S = \{x : w(x) \neq 0\},$$

then u is identically zero (this holds, for example, if $S = \Omega$ or $p = 2$ and S contains an open set). Then (3.2) remains valid, and hence so does the proof of Theorem 3.1 (ii). For another unique continuation property for $p \geq 2$, see [L].

Corollary 3.6. *If $w = 0$ a.e. on a nonempty open set $\Omega^+ \subset \Omega$, and $0 > \mu^+$ of (2.1), then no solution of (3.1) exists with u of one sign.*

Proof. Theorem 2.2 (b) implies that $\mu_1(\lambda) \leq \mu^+ < 0$. It then follows that $\mu_1(\lambda) = 0$ has no solution. $\square$

Remark 3.7. The condition that $0 > \mu^+$ is realized if q is sufficiently negative on Ω^+ .

4. RIGHT INDEFINITE CASES

Here we assume $\pm w > 0$ on positive measure sets $S_{\pm}$. We start with the case which is called (left) semidefinitizable in [AM] in the case $p = 2$.

Theorem 4.1. *If $\mu_1(\lambda_0) = 0$ but $\mu_1(\lambda) > 0$ for no λ , then any eigenfunction u_0 corresponding to λ_0 satisfies $\int_{\Omega} w|u_0|^p = 0$ and for any other eigenpair (λ, u) , (1.4) holds.*

Proof. By Theorem 2.1 (b), $0 = \mu'_1(\lambda_0) = -\int_{\Omega} w|u_0|^p$. By (3.1), if λ is another eigenvalue, then $I(\lambda, u) = 0$ and $I(\lambda_0, u) > \mu_1(\lambda_0) = 0$, whence $(\lambda - \lambda_0) \int_{\Omega} w|u|^p = I(\lambda_0, u) > 0$, unless u is proportional to u_0 . But (3.1) and $u = cu_0$ yield

$$(4.1) \quad (\lambda - \lambda_0)w[u_0]_p = 0 \quad \text{on } \Omega,$$

which contradicts the facts that $\lambda \neq \lambda_0$, $w \not\equiv 0$ on Ω , and u_0 does not vanish on Ω . $\square$

We turn now to the (left definitizable) case outlined in the introduction.

Theorem 4.2. *The following are equivalent. (i) $\mu_1(\lambda_0) > 0$ for some λ_0 (i.e., $I(\lambda_0, u) > 0$ for some λ_0 and all u). (ii) Two eigenvalues $\lambda_1^{\pm}$ exist with positive eigenfunctions, where $\lambda_1^- < \lambda_0 < \lambda_1^+$. Moreover, any eigenpair (λ, u) then satisfies (1.4).*

Proof. By Theorem 2.2 and the analogue for $\text{ess inf } w$, $\mu_1(\lambda) \rightarrow -\infty$ as $\lambda \rightarrow \pm\infty$. By Theorem 2.1 (b) and the fact that $\mu_1(\lambda_0) > 0$, μ_1 has two zeros $\lambda_1^{\pm}$ as required. This proves that (i) implies (ii).

If (λ, u) satisfy (3.1), then $I(\lambda, u) = 0$. On the other hand $I(\lambda_0, u) \geq \mu_1(\lambda_0)$, so we again obtain

$$(\lambda - \lambda_0) \int_{\Omega} w|u|^p = I(\lambda_0, u) > 0.$$

So (i) implies (1.4).

If $\sup_{\lambda} \mu_1(\lambda) < 0$, then $\mu_1(\lambda) = 0$ has no solution. If $\sup_{\lambda} \mu_1(\lambda) = 0$, then $\mu_1(\lambda) = 0$ has just one solution by Theorem 4.1. So (i) follows from (ii). $\square$

Remark 4.3. Replacing (3.1) by $-\Delta_p u + (q - \lambda w)[u]_p = \mu[u]_p$ for small $\mu < 0$ we convert the situation of Theorem 4.1 to that of Theorem 4.2, and there are two positive eigenfunctions $u_1^{\pm}(\mu)$ corresponding to $\lambda_1^{\pm}$. We have linear independence of $u_1^{\pm}$ by simple computation (cf. (4.1)), so $S(\mu) := \text{sp}\{u_1^{\pm}(\mu)\}$ is two dimensional. Suppose $p = 2$. Then as $\mu \rightarrow 0$, $S(\mu)$ becomes the two-dimensional algebraic eigenspace spanned by the eigenvector and a root (or “generalised”) eigenvector (cf. [BB2]). The former is a “ghost state”, according to [AM], although perhaps that term is more appropriate for the latter. When $p \neq 2$, $u_1^{\pm}(\mu)$ still tend to an eigenvector as $\mu \rightarrow 0$, but it is not known whether a root vector still exists.

Remark 4.4. The results of this section contain those for the case $p = 2$ given in [AM]. We also mention that some special cases of (1.1) with either Dirichlet or Neumann condition have been studied by Hess and Kato [HK] and Senn and Hess [SH] for the case $p = 2$ and by the authors in [BH1–2] and [H].

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