PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 6, June 1995

ε-ISOMETRIC EMBEDDINGS

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(Communicated by Dale Alspach)

ABSTRACT. In this paper we study into ε-isometries. We prove that if φ is an ε-isometry from $L^p(\Omega_1, \Sigma_1, \mu_1)$ into $L^p(\Omega_2, \Sigma_2, \mu_2)$ (for some p, $1 < p < \infty$), then there is a linear operator $T : L^p(\Omega_2, \Sigma_2, \mu_2) \mapsto L^p(\Omega_1, \sigma_1, \mu_1)$ with $\|T\| = 1$ such that $\|T \circ \varphi(f) - f\| \leq 6\varepsilon$ for each $f \in L^p(\Omega_1, \sum_1, \mu_1)$. This forms a link between an into isometry result of Figiel and a surjective ε-isometry result of Gevirtz in the case of L^p spaces.

1. INTRODUCTION

Let X and Y be real Banach spaces. For a fixed $\varepsilon > 0$, a map $\varphi : X \mapsto Y$ is called an ε-isometry if

$$|\,\|\varphi(x) - \varphi(y)\| - \|x - y\|\,| \leq \varepsilon \quad \text{for all } x, y \in X.$$

In 1945, Hyers and Ulam [7] raised the following question:

> Does there exist a constant $M > 0$ depending only on X and Y with the following property: For each $\varepsilon > 0$ and each surjective ε-isometry $\varphi : X \mapsto Y$ there is an isometry $\Phi : X \mapsto Y$ with $\|\varphi(x) - \Phi(x)\| \leq M\varepsilon$ for each $x \in X$?

In 1983, Gevirtz [5], based on a body of previous partial results extending over 38 years, proved that the Hyers-Ulam question has a positive answer with $M = 5$ independent of X and Y. (For history on this topic, see [3].) Note that the assumption that the map φ be surjective is necessary (cf. [7]).

In 1967, Figiel [4] proved the following non-surjective substitute for the Mazur-Ulam theorem [10]: for any isometry Φ from X into Y with $\Phi(0) = 0$, there is a linear operator T of norm one from $\overline{\text{span}}\Phi(X)$ onto X such that $T \circ \Phi$ is the identity on X. In light of Figiel's result, it is tempting to reformulate the Hyers-Ulam ε-isometry problem for into maps as:

(*) Does there exist a constant $M > 0$ depending only on X and Y with the following property: For each $\varepsilon > 0$ and each into ε-isometry $\varphi : X \mapsto Y$ with $\varphi(0) = 0$ there is a continuous linear operator T from $\overline{\text{span}}\varphi(X)$ onto X such that $\|T \circ \varphi(x) - x\| \leq M\varepsilon$ for each $x \in X$?

Received by the editors September 28, 1993.
1991 *Mathematics Subject Classification*. Primary 46B99.
Key words and phrases. ε-isometry, Banach space, linear operator, isometry.

As the following example shows, however, such an ε-isometry reformulation is doomed in general.

Example 1. Fix $\varepsilon > 0$. Let X be an uncomplemented subspace of some separable Banach space Y. Let φ_0 be a 1-1 map from X onto $B(Y)$, the closed unit ball of Y, with $\varphi_0(0) = 0$. Define a map $\varphi : X \mapsto Y$ by $\varphi(x) = x + \frac{\varepsilon}{2} \cdot \varphi_0(x)$ for each $x \in X$. Then φ is an ε-isometry with $\varphi(0) = 0$ and with $\overline{\operatorname{span}}\varphi(X) = Y$.

Suppose that there are an $M > 0$ and a continuous linear operator T from $\overline{\operatorname{span}}\varphi(X)$ onto X such that $\|T \circ \varphi(x) - x\| \le M\varepsilon$ for each $x \in X$. Then $\|Tx + \frac{\varepsilon}{2} T \circ \varphi_0(x) - x\| \le M\varepsilon$ and hence $\|T(nx) - nx\| \le M\varepsilon + \frac{\varepsilon}{2}\|T\|$ for each x in X and each positive integer n. Dividing by n and letting $n \to \infty$ yields that $Tx = x$ for each $x \in X$. It follows that T is a continuous linear projection from Y onto X, which is impossible by choice of X.

Observe that $\varphi - \varphi_0$ is linear in the above example and hence is nicely behaved, while φ_0 itself is uncontrolled. As we shall see below, under a variety of special conditions on X and Y, each into ε-isometry may be decomposed into a nicely behaved portion (corresponding to $\varphi - \varphi_0$ above) and one (corresponding to φ_0) whose deleterious effects may be successfully limited, leading ultimately to a positive answer to $(*)$ in such cases.

Throughout this paper, $\mathscr{U}$ always denotes a fixed nontrivial ultrafilter on the set $\mathscr{N}$ of positive integers. $\mathfrak{R}$ denotes the real line.

Say that $P : Y \mapsto Z$ is a CS projection (contractive surjective projection) provided P is a linear projection of norm one from Y onto the subspace Z of Y.

2. Preliminaries

We first start with a generalization of a result of Figiel [4] which gives an affirmative answer to the question $(*)$ raised in §1 when the domain is one dimensional.

Lemma 2. *If $\varphi : \mathfrak{R} \mapsto Y$ is an into ε-isometry with $\varphi(0) = 0$, then there is an $F \in Y^*$ with $\|F\| = 1$ such that*

$$|F \circ \varphi(t) - t| \le 5\varepsilon \quad \textit{for each } t \in \mathfrak{R}.$$

Proof. We follow Figiel's idea. Let n be a positive integer. The Hahn-Banach theorem guarantees the existence of an $F_n \in Y^*$ with $\|F_n\| = 1$ such that

$$F_n(\varphi(n) - \varphi(-n)) = \|\varphi(n) - \varphi(-n)\|.$$

For every $t \in [-n, n]$,

$$\begin{aligned} 2n + 2\varepsilon &\ge \|\varphi(t) - \varphi(-n)\| + \|\varphi(n) - \varphi(t)\| \\ &\ge F_n(\varphi(t) - \varphi(-n)) + \|\varphi(n) - \varphi(t)\| \\ &\ge F_n(\varphi(t) - \varphi(-n)) + F_n(\varphi(n) - \varphi(t)) \\ &= F_n(\varphi(n) - \varphi(-n)) = \|\varphi(n) - \varphi(-n)\| \ge 2n - \varepsilon. \end{aligned}$$

It follows that

$$t + n - 4\varepsilon \le F_n(\varphi(t) - \varphi(-n)) \le t + n + \varepsilon. \tag{1}$$

When $t = 0$, (1) becomes $n - 4\varepsilon \le F_n(-\varphi(-n)) \le n + \varepsilon$, and thus the general form of (1) may be written

$$t - 5\varepsilon \le F_n(\varphi(t)) \le t + 5\varepsilon \quad \text{for each } t \in [-n, n]. \tag{2}$$

Let $x = \sum_{i=1}^{k} a_i \varphi(t_i) \in \operatorname{span} \varphi(\mathfrak{R})$ and $m = \max\{|t_i| : 1 \le i \le k\}$. If $n \ge m$, then from (2)

$$-\sum_{i=1}^{k} |a_i|(m + 5\varepsilon) \le F_n(x) \le \sum_{i=1}^{k} |a_i|(m + 5\varepsilon).$$

Therefore for each fixed $x \in \operatorname{span} \varphi(\mathfrak{R})$, $\{F_n(x)\}_{n=1}^{\infty}$ is bounded. It follows that $\overline{F}(x) = \lim_{\mathscr{U}} F_n(x)$ exists for each $x \in \operatorname{span} \varphi(\mathfrak{R})$, and $\overline{F}$ is a linear functional defined on $\operatorname{span} \varphi(\mathfrak{R})$ with $\|\overline{F}\| = 1$. Moreover, for each $t \in \mathfrak{R}$,

$$t - 5\varepsilon \le \overline{F}(\varphi(t)) \le t + 5\varepsilon.$$

Thus any norm-preserving extension $F \in Y^*$ of $\overline{F}$ has the properties:

(i) $\|F\| = 1$; and

(ii) $|F \circ \varphi(t) - t| \le 5\varepsilon$ for each $t \in \mathfrak{R}$.

This completes the proof.

One candidate isometric approximation of a given ε-isometry is the ultrafilter limit of that ε-isometry as defined below.

Definition 3. Given a map $\varphi : X \mapsto Y$ and a subsequence $\{k(n)\}$ of $\{n\}$, define a map $\hat{\varphi} : X \mapsto Y$ provided the following limit exists for each $x \in X$:

$$\hat{\varphi}(x) = \lim_{\mathscr{U}} \frac{\varphi(k(n)x)}{k(n)} \quad \text{for each } x \in X.$$

(Note that the possible dependence of $\hat{\varphi}$ on $\{k(n)\}$ is suppressed. This will never cause confusion in the sequel.)

Property (Q) defined below is the key to controlling the "badly behaved" part of an into ε-isometry.

Definition 4. Let $\varphi : X \mapsto Y$ be a map. Then φ has Property (Q) provided

(i) $\varphi(0) = 0$;

(ii) there is a fixed subsequence $\{k(n)\}$ of $\{n\}$ such that $\hat{\varphi} : X \mapsto Y$ as defined in Definition 3 exists; and

(iii) if we let $Z = \overline{\operatorname{span}} \hat{\varphi}(X)$, then there is a CS projection $P_Z : Y \mapsto Z$.

3. Main results

There are several choices for X and Y for which a positive answer to the question $(*)$ stated in §1 is obtained. Each is a consequence of the following theorem.

Theorem 5. *Let Y be a smooth Banach space. Let $\varphi : X \mapsto Y$ be an into ε-isometry with Property* (Q). *Set $Z = \overline{\operatorname{span}} \hat{\varphi}(X)$, and denote by $P_Z : Y \mapsto Z$ the associated CS projection guaranteed by Property* (Q). *Suppose further that there is a closed linear subspace $X' \subset Z$ with the following properties:*

(i) *There is a CS projection $P_{X'} : Z \mapsto X'$; and*

(ii) *X' is linearly isometric to X under the map $P_{X'} \circ \hat{\varphi}$.*

Then there is a linear operator $T : Y \mapsto X$ *with* $\|T\| = 1$ *such that*

$$\|T \circ \varphi(x) - x\| \le 6\varepsilon \quad \text{for each } x \in X.$$

Proof. If X is one dimensional, Lemma 2 provides an even stronger conclusion under weaker hypotheses. Thus assume $\dim X \ge 2$ in the remainder of the proof.

Let $U = P_{X'} \circ \hat{\varphi}$. Thus $U : X \mapsto X'$ is a linear isometry by assumption. Let $(X')^{\perp} = (I - P_{X'})(Z)$ and $Z^{\perp} = (I - P_Z)(Y)$. Then $Z = X' \oplus (X')^{\perp}$ and $Y = X' \oplus (X')^{\perp} \oplus Z^{\perp} = Z \oplus Z^{\perp}$. Thus it is possible to uniquely decompose φ as $\varphi(x) = \varphi_{X'}(x) + \varphi_{(X')^{\perp}}(x) + \varphi_{Z^{\perp}}(x)$ for each $x \in X$. Note that $\text{Range}(\hat{\varphi}) \subset Z$ guarantees that $\widehat{\varphi_{Z^{\perp}}} = 0$, and hence $\hat{\varphi} = \widehat{\varphi_{X'}} + \widehat{\varphi_{(X')^{\perp}}}$. Moreover, $U = \widehat{\varphi_{X'}}$ by the definition of U. It will be shown below that

$$\|\varphi_{X'}(x) - U(x)\| \le 6\varepsilon \quad \text{for each } x \in X. \tag{3}$$

Once done, the proof of Theorem 5 is direct. Indeed, assuming (3), let $T = U^{-1} \circ P_{X'} \circ P_Z$. Then $\|T\| = 1$ and for each $x \in X$,

$$\begin{aligned}\|T \circ \varphi(x) - x\| &= \|U^{-1} \circ P_{X'} \circ P_Z \circ \varphi(x) - x\| \\ &= \|P_{X'} \circ P_Z \circ \varphi(x) - U(x)\| = \|\varphi_{X'}(x) - U(x)\| \le 6\varepsilon,\end{aligned}$$

which establishes the theorem. The remainder of the argument is given to a proof of (3).

Define $\psi : X' \mapsto Y$ by $\psi(y) = \varphi(U^{-1}(y))$ for each $y \in X'$. Since U is an isometry, ψ is an ε-isometry. Set $\psi_{X'} = P_{X'} \circ P_Z \circ \psi$, $\psi_{(X')^{\perp}} = (I - P_{X'}) \circ P_Z \circ \psi$, and $\psi_{(Z)^{\perp}} = (I - P_Z) \circ \psi$. Then both $\psi = \psi_{X'} + \psi_{(X')^{\perp}} + \psi_{Z^{\perp}}$ and $\hat{\psi} = \widehat{\psi_{X'}} + \widehat{\psi_{(X')^{\perp}}}$. Moreover, from the definition of ψ it follows that $\widehat{\psi_{X'}} = \text{id}_{X'}$.

Fix $x \in X$ and let $y = U(x)$. If $y = \psi_{X'}(y)$, then obviously $\|y - \psi_{X'}(y)\| \le 6\varepsilon$. The work comes in establishing this same inequality when $y \ne \psi_{X'}(y)$ and it helps to define z in this case by $z = (y - \psi_{X'}(y))/\|y - \psi_{X'}(y)\|$.

It follows from Lemma 2 that there is an $F \in Y^*$ such that $\|F\| = 1$ and

$$|F \circ \psi(tz) - t| \le 5\varepsilon \quad \text{for each } t \in \mathfrak{R}. \tag{4}$$

Hence $F \circ \hat{\psi}(tz) = t$ for each $t \in \mathfrak{R}$. Now write

$$\hat{\psi}(tz) = \hat{\psi}_{X'}(tz) + \widehat{\psi_{(X')^{\perp}}}(tz) = tz + \widehat{\psi_{(X')^{\perp}}}(tz)$$

for each $t \in \mathfrak{R}$. Let $G \in (\text{span}\{z\} \oplus (X')^{\perp} \oplus Z^{\perp})^*$ be defined by

$$G(sz + z_2) = s \quad \text{for each } s \in \mathfrak{R} \text{ and } z_2 \in (X')^{\perp} \oplus Z^{\perp}.$$

Then $\|G\| = 1$ and $G \circ \hat{\psi}(z) = 1$. Since $\hat{\psi}(z)$ is a smooth point in Y, we have $F = G$ on $\text{span}\{z\} \oplus (X')^{\perp} \oplus Z^{\perp}$. In particular, then, $F(z) = 1$ and $(X')^{\perp} \oplus Z^{\perp} \subseteq F^{-1}(0)$. By (4), for each $t \in \mathfrak{R}$,

$$\begin{aligned}|F \circ \psi_{X'}(tz) - t| &= |F \circ \psi_{X'}(tz) + F \circ \psi_{(X')^{\perp}}(tz) + F \circ \psi_{Z^{\perp}}(tz) - t| \\ &= |F \circ \psi(tz) - t| \le 5\varepsilon.\end{aligned}$$

In the rest of the proof, we make use of some ideas in [6]. Choose $u \in F^{-1}(0) \cap X'$ with $\|u\| = 1$ such that $\{y, \psi_{X'}(y)\} \subset \text{span}\{z, u\}$. Hence, there exist numbers β and γ such that $\psi_{X'}(y) = \beta z + \gamma u$. Thus

$$\begin{aligned}\|tz - y\| &\ge \|\psi(tz) - \psi(y)\| - \varepsilon \ge F \circ P_{X'}(\psi(tz) - \psi(y)) - \varepsilon \\ &= F \circ \psi_{X'}(tz) - F \circ \psi_{X'}(y) - \varepsilon \ge t - 5\varepsilon - \beta - \varepsilon \\ &= t - \beta - 6\varepsilon.\end{aligned} \tag{5}$$

Let $\alpha = \|y - \psi_{X'}(y)\| + \beta$. Then $y = \alpha z + \gamma u$. Since z is a smooth point with support functional F, $\|\cdot\|$ is Gateaux differentiable at z with the Gateaux derivative F. Thus $\|z + tu\| = 1 + o(t)$ as $t \to 0$. Hence,

$$\|tz - y\| = \|tz - \alpha z - \gamma u\| = (t - \alpha) + o(1) \quad \text{as } t \to \infty. \tag{6}$$

Combining (5) and (6) gives

$$\alpha - \beta \leq 6\varepsilon. \tag{7}$$

Because $z = (y - \psi_{X'}(y))/\|y - \psi_{X'}(y)\|$ and $F(z) = 1$, it follows that

$$F(y) - F \circ \psi_{X'}(y) = \|y - \psi_{X'}(y)\|.$$

But from the description of $\psi_{X'}(y)$ and y in terms of z and u it is immediate that

$$F(y) - F \circ \psi_{X'}(y) = \alpha - \beta.$$

Hence (7) implies that

$$\|y - \psi_{X'}(y)\| \leq 6\varepsilon.$$

That is,

$$\|U(x) - \varphi_{X'}(x)\| \leq 6\varepsilon,$$

as was to be shown. This completes the proof.

The following theorem is an immediate consequence of Theorem 5.

Theorem 6. *Let Y be strictly convex and smooth. Suppose that $\varphi : X \mapsto Y$ is an into ε-isometry satisfying Property* (Q). *Then there is a continuous linear operator T from Y onto X such that $\|T\| = 1$ and*

$$\|T \circ \varphi(x) - x\| \leq 6\varepsilon \quad \textit{for each } x \in X.$$

Proof. Note that $\hat{\psi}$ is a linear isometry since Y is strictly convex. Hence $Z = \overline{\operatorname{span}}\hat{\varphi}(X) = \hat{\psi}(X)$. Now the identity map $\mathrm{id}_Z : Z \mapsto Z$ plays the role of $P_{X'}$ in Theorem 5, which may be applied directly to complete the proof.

Specializing Theorem 6 to the L^p spaces leads to the next result.

Theorem 7. *Let $(\Omega_i, \Sigma_i, \mu_i)$, $i = 1, 2$, be any two measure spaces, and fix p, $1 < p < \infty$. Assume that φ is an ε-isometry of $L^p(\Omega_1, \Sigma_1, \mu_1)$ into $L^p(\Omega_2, \Sigma_2, \mu_2)$ with $\varphi(0) = 0$. Then there is a continuous linear operator $T : L^p(\Omega_2, \Sigma_2, \mu_2) \mapsto L^p(\Omega_1, \Sigma_1, \mu_1)$ with $\|T\| = 1$ such that*

$$\|T \circ \varphi(x) - x\| \leq 6\varepsilon \quad \textit{for each } x \in L^p(\Omega_1, \Sigma_1, \mu_1).$$

Proof. First we need to recall the following two results:

(I) [8, p. 162] Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces and $1 < p < \infty$. Suppose that Y is a closed subspace of $L^p(\Omega_2, \Sigma_2, \mu_2)$ which is linearly isometric to $L^p(\Omega_1, \Sigma_1, \mu_1)$. Then there is a CS projection of $L^p(\Omega_2, \Sigma_2, \mu_2)$ onto Y.

(II) (Bourgin [2]) Fix p, $1 < p < \infty$, and let φ be an ε-isometry of X into $L^p(\Omega, \Sigma, \mu)$ with $\varphi(0) = 0$. Then $\hat{\varphi}(x) = \lim_{n\to\infty} \varphi(2^n x)/2^n$ exists for each $x \in X$ and $\hat{\varphi}$ is a linear isometry. (The special case $p = 2$ was established by Hyers and Ulam [7].)

By (I) and (II) above, $\hat{\varphi}(x) = \lim_{n\to\infty} \varphi(2^n x)/2^n$ exists for each $x \in L^p(\Omega_1, \Sigma_1, \mu_1)$, and $\hat{\varphi}(L^p(\Omega_1, \Sigma_1, \mu_1))$ is the range of a CS projection on $L^p(\Omega_2, \Sigma_2, \mu_2)$. Hence the theorem follows from Theorem 6.

The next theorem is concerned with ε-isometries whose ranges are Hilbert spaces.

Theorem 8. *Let $\mathscr{H}$ be a Hilbert space and X any Banach space. Let $\varphi : X \mapsto \mathscr{H}$ be an into ε-isometry with $\varphi(0) = 0$. Then there is a continuous linear operator $T : \mathscr{H} \mapsto X$ with $\|T\| = 1$ such that*

$$\|T \circ \varphi(x) - x\| \leq 6\varepsilon \quad \textit{for each } x \in X.$$

The last result involves ε-isometries with Property (Q) whose ranges have the RNP.

Theorem 9. *Let X and Y be Banach spaces with X separable and Y both smooth and with the RNP. Suppose that $\varphi : X \mapsto Y$ is an into ε-isometry satisfying property* (Q). *Then there is a linear operator $T : X \mapsto Y$ with $\|T\| = 1$ such that*

$$\|T \circ \varphi(x) - x\| \leq 6\varepsilon \quad \textit{for each } x \in X.$$

Proof. The map $\hat{\varphi}$ is an isometry of X into Y. By Figiel's theorem [4], there is a norm-one linear operator $T : Z = \overline{\operatorname{span}}\hat{\varphi}(X) \mapsto X$ such that $T \circ \hat{\varphi} = \mathrm{id}_X$. Since Y has the RNP and X is separable, the operator $D\hat{\varphi}(a)$ exists for some $a \in X$ ([9] or [1]). Then $D\hat{\varphi}(a)$ is a linear isometry of X onto the subspace $X' = [D\hat{\varphi}(a)](X) \subset Y$. Define $P_{X'} = D\hat{\varphi}(a) \circ T$. Then $P_{X'}$ is a CS projection of Z onto X' and $P_{X'} \circ \hat{\varphi} = D\hat{\varphi}(a)$. Thus an application of Theorem 5 completes the proof.

Remark 10. Separability of X guarantees that $\hat{\varphi}$ is Gateaux differentiable at some point in X. It is unknown whether the theorem remains true without separability of X.

Not surprisingly, when finite-dimensional Banach spaces were considered, correspondingly stronger results may be established. Indeed, let Y be a finite-dimensional Banach space and $\varphi : X \mapsto Y$ an into ε-isometry. Then $\hat{\varphi}(x) = \lim_{\mathscr{U}} \frac{\varphi(nx)}{n}$ exists for each $x \in X$. As a consequence of this observation, the following result is a special case of Theorem 9.

Corollary 11. *Let Y be a smooth finite-dimensional Banach space. Suppose that $\varphi : X \mapsto Y$ is an into ε-isometry with $\varphi(0) = 0$ for which there is a CS projection of Y onto* $\operatorname{span} \hat{\varphi}(X)$. *Then there is a linear operator $T : X \mapsto Y$ with $\|T\| = 1$ such that*

$$\|T \circ \varphi(x) - x\| \leq 6\varepsilon \quad \textit{for each } x \in X.$$

Acknowledgment

This paper is part of the author's Ph.D. thesis prepared at the Howard University under the supervision of Professor Richard D. Bourgin. The author wishes to thank him for his great guidance and help.

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