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# CONTRACTIVE PROJECTIONS IN NONATOMIC FUNCTION SPACES

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ABSTRACT. We prove that there are no 1-complemented subspaces of finite codimension in separable rearrangement-invariant nonatomic function spaces not isometric to $L_2$.

We study contractive projections onto finite-codimensional subspaces of real nonatomic function spaces. In general such projections are not common. It is well known that only in Hilbert space there exists a contractive projection onto every subspace of fixed finite codimension (cf. [1]).

The study of contractive projections (and more general projections with minimal norm) is important in approximation theory (cf. the survey of Cheney and Price [3]).

It is known that there are no 1-complemented subspaces of finite-codimension in $C[0, 1]$ (Wulbert [13]) and in $L_1(\mu)$ if the measure $\mu$ is nonatomic ([6, Corollary IV.1.15], [4]). De Figueiredo and Karlovitz [5] (1970) proved that if $\mu$ is nonatomic, then there are no 1-complemented hyperplanes in $L_p(\Omega, \mu)$, $1 < p < \infty$, $p \neq 2$ (cf. also [2]).

In this paper we prove that in rearrangement-invariant nonatomic function spaces not isometric to $L_2$ there are no 1-complemented subspaces of any finite codimension.

We use the terminology and notation as in [9].

Our method of proof is surprisingly simple—it is based on the following observation:

**Proposition 1** (cf. [7, 11]). *In a real Banach space $X$ if $P$ is a projection, then $\|I - P\| = 1$ (where $I$ denotes identity operator) if and only if $x^*(Px) \geq 0$ for all $x \in X$ and $x^* \in X^*$ norming for $x$.*

In [7, Theorem 4.3] (cf. [8, 10]) Kalton and the author proved the nonexistence of 1-complemented hyperplanes in a wide class of nonatomic function spaces. However, the original theorem uses special technical phraseology, so we state it below in the language of projections:

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**Theorem 2.** *Suppose $X$ is a real order-continuous Köthe function space on $(\Omega, \mu)$ and $\mu$ is nonatomic. Then the hyperplane $H$ in $X$ is* 1*-complemented if and only if there exists a nonnegative measurable function $\omega$ with* $\operatorname{supp} w = B = \operatorname{supp} f$, *where $f \in H^{\perp} \subset X^*$, so that for any $x \in X$ with* $\operatorname{supp} x \subset B$

$$\|x\| = \left(\int |x|^2 w \, d\mu\right)^{1/2}.$$

Hence there are no 1-complemented hyperplanes in $X$ unless $L_2$ is isometric to a band in $X$. In particular, there are no 1-complemented hyperplanes in separable r.i. spaces on $[0, 1]$ [7, Theorem 4.4].

As part of the proof of Theorem 2 we proved the following fact, which we state separately for the future use.

**Proposition 3** ([10, Proposition 2.8]). *Let $X$ be an order-continuous Köthe function space on $(\Omega, \mu)$, where $\mu$ is nonatomic. Suppose that the set*

$$\Lambda = \left\{\frac{x^*}{x} : x \in X\,,\ x^* \in X^* \text{ norming for } x\right\}$$

*is one-dimensional, i.e., $\Lambda \subset \{aw : a \in \mathbb{R}\}$ for some $w \in L_0(\Omega, \mu)$. Then $X$ is isometric to $L_2(w\, d\mu)$.*

Now we are ready to prove our main result.

**Theorem 4.** *Suppose $\mu$ is nonatomic and $X$ is a separable r.i. space on $([0, 1], \mu)$ not isometric to $L_2$. Then there are no* 1*-complemented subspaces of any finite codimension in $X$.*

For the proof we need the following measure-theoretic lemma.

**Lemma 5.** *Suppose $\mu$ is nonatomic, and suppose $f_1, \dots, f_n, g_1, \dots, g_n \in L_1(\mu)$ are such that $g_1, \dots, g_n$ are linearly independent and*

$$\sum_{i=1}^{n} \left(\int h f_j \, d\mu\right) \left(\int h g_j \, d\mu\right) \geq 0 \tag{1}$$

*whenever $|h| = 1$ a.e. Then $\{f_j\}_{j=1}^n \subset \operatorname{span}\{g_j\}_{j=1}^n$.*

*Proof.* Consider the operator $T \colon \mathbb{R}^{2n} \to L_1(\mu)$ defined by

$$T(a_1, b_1, \dots, a_n, b_n) = \sum_{i=1}^{n} a_i f_i + b_i g_i .$$

To prove the lemma it is enough to show that the dimension of the range of $T$ is equal to $n$, since $f_1, \dots, f_n$, $g_1, \dots, g_n$ are contained in the range and $g_1, \dots, g_n$ are linearly independent. For that it is enough to show that $\dim T^*(L_\infty(\mu)) \leq n$. Notice that the operator $T^*$ is defined by $T^*h = (\int hf_1\, d\mu, \int hg_1\, d\mu, \dots, \int hf_n\, d\mu, \int hg_n\, d\mu)$ for $h \in L_\infty(\mu)$, and let $\Gamma = \{T^*h \colon |h| = 1 \text{ a.e.}\}$.

Clearly $T^*(L_\infty(\mu)) = \operatorname{span}\Gamma$, and we immediately see that $\Gamma = -\Gamma$. Now consider the $\mathbb{R}^{2n}$-valued measure $\overline{\mu}$: $\overline{\mu}(A) = (\int \chi_A f_1\, d\mu, \int \chi_A g_1\, d\mu, \dots, \int \chi_A f_n\, d\mu, \int \chi_A g_n\, d\mu)$. By Liapunoff's theorem [12] the angle $\mathscr{R}(\overline{\mu})$ of $\overline{\mu}$ is convex and thus also $\Gamma$ is convex $(\Gamma = T^*1 - 2\mathscr{R}(\overline{\mu}))$. Therefore inequality (1) yields that

$$(2) \qquad \sum_{j=1}^{n} s_j t_j \geq 0 \quad \text{for every } (s_1, t_1, \dots, s_n, t_n) \in T^*(L_\infty(\mu)).$$

To finish the proof consider $V = \operatorname{span}\{e_{2k-1} - e_{2k} : k = 1, \dots, n\} \subset \mathbb{R}^{2n}$, where $e_j$ denotes the natural basis of $\mathbb{R}^{2n}$. By (2) $V \cap T^*(L_\infty(\mu)) = \{0\}$, so $\dim T^*(L_\infty(\mu)) \leq n$. $\square$

*Proof of Theorem* 4. Suppose that $F$ is a closed linear subspace of codimension $n$ in $X$, and let $u_1, \dots, u_n$ be linearly independent functions in $X$ such that $\operatorname{span}\{F, u_1, \dots, u_n\} = X$. Denote $P: X \to F$ a contractive projection onto $F$, and consider $Q = I - P$. Then $Q = \sum_{j=1}^{n} f_j \otimes u_j$ for some linearly independent $f_1, \dots, f_n \in X^*$.

By Proposition 1 for any $x \in X$ and $x^* \in X^*$ norming for $x^*(Qx) \geq 0$. Next for any $h$ with $|h| = 1$ a.e. $hx^*$ is norming for $hx$ if $x^*$ is norming for $x$. Hence $hx^*(Q(hx)) \geq 0$, i.e.,

$$\sum_{j=1}^{n} \left( \int f_j hx \, d\mu \right) \left( \int u_j hx^* \, d\mu \right) \geq 0.$$

By Lemma 5 $f_1 x \in \operatorname{span}\{u_j x^* : j = 1, \dots, n\}$. So if $B = \operatorname{supp} f_1 (\mu(B) > 0)$, then $\frac{x}{x^*}|_B \in \operatorname{span}\{\frac{u_j}{f_1} : j = 1, \dots, n\}$. By re-arrangement invariance of $X$ for every measure-preserving map $\sigma: [0, 1] \to [0, 1]$ $x^* \circ \sigma$ is norming for $x \circ \sigma$ and so

$$\frac{x \circ \sigma}{x^* \circ \sigma}\Big|_B \in \operatorname{span}\left\{ \frac{u_j}{f_1} : j = 1, \dots, n \right\},$$

i.e., the set $\{[(\frac{x}{x^*}) \circ \sigma]\big|_B | \sigma : [0, 1] \to [0, 1]$ measure preserving$\}$ is finite dimensional, which is impossible unless $\frac{x}{x^*}$ is a constant. But then by Proposition 3 we conclude that $X$ is isometric to $L_2[0, 1]$, contrary to our assumption. $\square$

*Remark.* Notice that in the proof of Theorem 4 we use re-arrangement invariance of $X$ only in the final step to conclude that if the set $\{\frac{x}{x^*} : x \in X, x^* \in X^*$ norming for $x\}$ is finite-dimensional, then it is one-dimensional. A similar conclusion is true also in spaces of the form $X(Y)$, $X_1(X_2(\dots(X_m)\dots))$, where $X$, $Y$, $X_j$ are r.i. and Theorem 4 holds as stated also for those spaces.


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