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NORMALIZING ELEMENTS IN *PI* RINGS

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ABSTRACT. This paper explores the question: If R is a prime PI ring and a an element such that $aR \supseteq Ra$, is it true that $aR = Ra$?

1. INTRODUCTION

In [M] Susan Montgomery raised the following question, which arose in connection with Picard groups:

Question. Let R be a prime PI ring and a an element of R such that aR is a (two-sided) ideal. Is it true that a is a normalizing element; i.e., that $aR = Ra$?

Indeed [M, Lemma 2] shows, without the PI hypothesis, that this holds provided that a is right regular and aR is an invertible R-ideal in the Martindale quotient ring of R.

In this paper we present some results, both positive and negative, related to this question.

2. EXAMPLES

First we make the elementary observation that the restriction to the case of a prime ring is essential. For if we let R be the tiled ring

$$R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix} \quad \text{and} \quad a = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

then one checks immediately that

$$aR = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 2\mathbb{Z} \end{pmatrix} \supsetneqq Ra = \begin{pmatrix} \mathbb{Z} & 2\mathbb{Z} \\ 0 & 2\mathbb{Z} \end{pmatrix}.$$

More seriously, we next show that, in general, the question has a negative answer even for an affine algebra:

Theorem 1. *There exists a prime PI ring R which is an affine* (*i.e., finitely generated*) *algebra over a field and which has an element a such that aR is an ideal but a is not normalizing.*

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Proof. Let k be any field, T be the commutative k-algebra $T = k[x, y, y^{-1}]$, and S be the tiled 2×2 matrix ring

$$S = \begin{pmatrix} k[y] + xT & T \\ xT & T \end{pmatrix}.$$

We will make use of the ideal $I = \begin{pmatrix} xT & T \\ xT & T \end{pmatrix}$ and the element $u = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}$. The following facts will be needed:

(i) S is an affine k-algebra, since T is affine; for S is generated over k by e_{22} times each of the generators of T, together with e_{11}, ye_{11}, e_{12} and xe_{21}.

(ii) I is indeed an ideal of S and is finitely generated, as both a right ideal and a left ideal by e_{12} and e_{22}.

(iii) $uI = Iu = I$.

(iv) $uS = Su = \begin{pmatrix} yk[y]+xT & T \\ xT & T \end{pmatrix} \neq S$.

Finally, let $R = \begin{pmatrix} S & S \\ I & S \end{pmatrix}$ and $a = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$. Since (i) and (ii) hold, one sees that R is an affine k-algebra. Also, using (iii) and (iv), one checks readily that

$$aR = \begin{pmatrix} S & S \\ I & uS \end{pmatrix} \supsetneqq Ra = \begin{pmatrix} S & uS \\ I & uS \end{pmatrix}.$$

3. Positive results

In this section we present two positive results together.

Theorem 2. *Let R be a prime PI ring which is finitely generated as a module over its center. If aR is an ideal of R, then $aR = Ra$.*

Proof. We may as well assume that $a \neq 0$. Then a is clearly a left regular element and, since a prime PI ring is a Goldie ring, one knows that a is regular. Moreover, R has a ring of quotients, Q say, in which a has an inverse. Thus, there is an automorphism μ of Q via $\mu(q) = a^{-1}qa$ for each q in Q. Of course, since aR is an ideal $Ra \subseteq aR$. Hence $a^{-1}Ra \subseteq R$ and so μ restricts to a ring endomorphism of R which is also an endomorphism of R as a module over its center, Z say. Moreover, if K is a quotient field of Z, then K is the center of Q and μ is a K-automorphism of Q; indeed $\mu = \ell(a^{-1})r(a)$, where $\ell()$ denotes left multiplication and $r()$ denotes right multiplication.

Since R is a finite module over Z, then Q is a finite dimensional K-algebra, say $[Q : K] = m = n^2$. Therefore we may view μ as an element of $M_m(K)$ and consider its determinant $\operatorname{Det}\mu$ and its characteristic polynomial $\chi_\mu(x)$. Of course, $\operatorname{Det}\mu = \operatorname{Det}(\ell(a^{-1}))\operatorname{Det}(r(a))$. If we let $N()$ denote the reduced norm, then it is known (see, for example, [C], pp. 274–275) that $(N(r(a)))^n = \operatorname{Det}(r(a))$ and $(N(\ell(a^{-1})))^n = \operatorname{Det}(\ell(a^{-1}))$. Hence $\operatorname{Det}\mu = 1$. In particular, we learn that the constant term of $\chi_\mu(x)$ is 1.

Next note that, since R is a finitely generated Z-module and μ is a Z-endomorphism of R, then μ is integral over Z. It follows that the coefficients of $\chi_\mu(x)$ are all integral over Z; i.e., belong to $\overline{Z}$, the integral closure of Z. [Why? Well, let μ satisfy the polynomial $p(x)$ in $Z[x]$ and let $m(x)$ denote the minimum polynomial of μ in $K[x]$. Then $m[x]$ divides $p(x)$, so all roots of $m(x)$ are integral over Z. This is also true, therefore, of the roots of $\chi_\mu(x)$. Hence the coefficients are, indeed, integral over Z.]

Suppose that $\chi_\mu(x) = x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + 1$. Then one sees that

$$\mu^{-1} = -(\mu^{n-1} + b_{n-2}\mu^{n-2} + \ldots + b_1\mu)$$

and so μ^{-1} belongs to $\overline{Z}[\mu]$. Hence μ^{-1} is integral over Z, from which it follows that μ^{-1} is in $Z[\mu]$ and so is a Z-endomorphism of R. Thus $aRa^{-1} \subseteq R$ and so $aR = Ra$.

Before giving a consequence of this, we note that, if R is a prime right Goldie ring with right quotient ring Q and aR is a nonzero ideal of R, then $R \subseteq aRa^{-1} \subseteq Q$.

Theorem 3. *Let R be a Noetherian prime PI ring which is affine over a field k (or, more generally, over a Noetherian subring of its center) and let aR be an ideal of R. Then $aR = Ra$.*

Proof. We make use of the trace ring TR of R; see, for example, [McR], Chapter 13, Section 9 and, in particular, 13.9.11 which shows that TR is finitely generated as an R-module and that TR is a finite module over its center. Bearing in mind that TR is generated, over R, by central elements, one sees that aTR is an ideal of TR. We can, therefore, see from Theorem 2 that $aTR = TRa$, and so $a^sRa^{-s} \subseteq TR$ for all s.

Now note that

$$aRa^{-1} \subseteq a^2Ra^{-2} \subseteq \ldots \subseteq a^sRa^{-s} \subseteq \ldots \subseteq TR.$$

Since $aR \supseteq Ra$, one sees that each of the subsets in the chain is a left R-submodule of the finitely generated R-module TR. Hence $a^sRa^{-s} = a^{s+1}Ra^{-s-1}$, for some s, and so $aR = Ra$.

We have recently learned that A. Braun has removed "affine" from the hypothesis of Theorem 3.

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