REFLEXIVITY OF TENSOR PRODUCTS
OF LINEAR TRANSFORMATIONS

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Let A and B be linear transformations on finite-dimensional Hilbert space. We characterize the reflexivity of $A \otimes B$ in terms of certain characteristics of A and B.

1. INTRODUCTION

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be separable, complex, Hilbert spaces, and suppose that T_1 and T_2 are bounded linear operators acting on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then the tensor product $T_1 \otimes T_2$ is a bounded linear operator acting on $\mathcal{H}_1 \otimes \mathcal{H}_2$, and one may ask to what degree the structure of $T_1 \otimes T_2$ is determined by that of T_1 and T_2. For instance, it was shown in [2] that the spectrum $\sigma(T_1 \otimes T_2)$ is determined by the equation

$$\sigma(T_1 \otimes T_2) = \sigma(T_1)\sigma(T_2) = \{\lambda\mu \colon \lambda \in \sigma(T_1), \ \mu \in \sigma(T_2)\}.$$

Also, if T_i is similar to U_i via the invertible operator X_i, $i = 1, 2$, then $T_1 \otimes T_2$ is similar to $U_1 \otimes U_2$ by virtue of the equation

$$(X_1 \otimes X_2)(T_1 \otimes T_2)(X_1 \otimes X_2)^{-1} = U_1 \otimes U_2,$$

and so the similarity invariants of $T_1 \otimes T_2$ are completely determined by those of T_1 and T_2. However, this information alone does not enable us to answer all questions concerning the structure of $T_1 \otimes T_2$ in terms of the structure of T_1 and T_2. For example, one may ask whether the reflexivity of $T_1 \otimes T_2$ follows from that of T_1 and (or) that of T_2. Very recently it was shown that in the infinite-dimensional case (see [5]) there exists T_1 and T_2 reflexive such that $T_1 \otimes T_2$ is not reflexive. One is led to ask whether this type of "pathology" is possible in the finite-dimensional case. We show here that it is not. We present in addition (Theorem 3.1) necessary and sufficient conditions for the reflexivity of the tensor product of two operators. This characterization of the tensor product is an easy consequence of the Deddens-Fillmore theorem [4] and a result from linear algebra concerning Jordan forms of tensor products.

Results concerning the Jordan form of tensor products were first obtained in the 1930s by Aitkin, Roth, and Littlewood [1, 8, 6]. Some gaps in the original proofs were filled in more recently by Marcus and Robinson [7] and by Brualdi [3]. These results were rediscovered independently by the authors. We present an elementary proof in §4.

2. NOTATION AND PRELIMINARIES

Let $\mathscr{B}(\mathscr{H})$ be the algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$, and let $\mathscr{A}$ be a subalgebra of $\mathscr{B}(\mathscr{H})$. We define $\mathrm{Lat}(\mathscr{A})$ to be the lattice of invariant subspaces of the family $\mathscr{A}$ and $\mathrm{Alg}(\mathrm{Lat}(\mathscr{A}))$ to be the subalgebra of $\mathscr{B}(\mathscr{H})$ of operators which leave invariant all elements of $\mathrm{Lat}(\mathscr{A})$. We say that $\mathscr{A}$ is a reflexive algebra if $\mathscr{A} = \mathrm{Alg}(\mathrm{Lat}(\mathscr{A}))$. We say that an operator $T \in \mathscr{B}(\mathscr{H})$ is reflexive if the weakly closed algebra generated by T and the identity operator I is reflexive.

Now, suppose that $\mathscr{H}$ is finite dimensional. For each λ in $\mathscr{C}$ and n in $\mathscr{N}$ (the set of positive integers) let $J(\lambda, n)$ be the $n \times n$ complex Jordan block matrix with eigenvalue λ (with 1's above the main diagonal). We denote by N_n the matrix $J(0, n)$, the nilpotent matrix with degree of nilpotence equal to n. Let θ be a fixed linear ordering of $\mathscr{C}$. An $m \times m$ complex matrix A will be said to be in *Jordan form* provided A is a direct sum of Jordan blocks, say $A = \bigoplus_{i=1}^{k} J(\lambda_i, n_i)$, in which the distinct terms of the sequence $\{\lambda_1, \ldots, \lambda_k\}$ appear as in the ordering θ and the Jordan blocks corresponding to the same eigenvalue appear with increasing size. In the following, we denote by $\mathscr{H}_n$ an n-dimensional complex Hilbert space equipped with a fixed (ordered) orthonormal basis $\varepsilon_n = \{e_1^n, \ldots, e_n^n\}$, and we say that an operator on $\mathscr{H}_n$ is in *Jordan form* when its matrix with respect to ε_n is in Jordan form. Clearly each operator T acting on $\mathscr{H}_n$ is similar to exactly one operator in Jordan form on $\mathscr{H}_n$ which we denote by $J(T)$.

Definition 2.1. Let $A \in \mathscr{B}(\mathscr{H})$, and let λ be an eigenvalue of A. We define:
- n_λ^A to be the size of the largest Jordan block belonging to λ,
- m_λ^A to be the size of the second largest Jordan block belonging to λ ($m_\lambda^A = 0$ if A has only one Jordan block belonging to λ),
- S_A to be the set of all nonzero eigenvalues of A associated with a one-dimensional eigenspace, and,
- T_A to be the set of all nonzero eigenvalues λ of A such that $n_\lambda^A - m_\lambda^A > 1$.

We say that λ has the *Deddens-Fillmore property for* A if $n_\lambda^A - m_\lambda^A \leq 1$.

We note that the Deddens-Fillmore theorem characterizes reflexive linear operators on finite-dimensional Hilbert space as exactly those operators A such that for each eigenvalue λ of A, $n_\lambda^A - m_\lambda^A \leq 1$. Thus, if A is reflexive, $T_A = \varnothing$.

Now, suppose that A and B are linear operators acting on the Hilbert spaces $\mathscr{H}_m$ and $\mathscr{H}_n$, respectively. In order to determine the reflexivity of $A \otimes B$, we need to know the relationship between the Jordan form of $A \otimes B$ and that of A and of B. Since $A \otimes B$ is similar to the matrix $J(A) \otimes J(B)$, we may assume that A and B are in Jordan form. Since, by definition, the matrices $J(A)$ and $J(B)$ relative to the orthonormal bases $\{e_k^m\}_{k=1}^m$ and $\{e_k^n\}_{k=1}^n$ for $\mathscr{H}_m$ and $\mathscr{H}_n$

have the form

$$J(\lambda_1, i_1) \oplus J(\lambda_2, i_2) \oplus \cdots \oplus J(\lambda_s, i_s)$$

and

$$J(\mu_1, j_1) \oplus J(\mu_2, j_2) \oplus \cdots \oplus J(\mu_t, j_t),$$

respectively, it is clear that the matrix of $J(A) \otimes J(B)$ with respect to the orthonormal basis $\{e_1^m \otimes e_1^n, e_1^m \otimes e_2^n, \ldots, e_1^m \otimes e_n^n, \ldots, e_m^m \otimes e_n^n\}$ will have the form

$$\bigoplus_{1 \le k \le s, \ 1 \le l \le t} (J(\lambda_k, i_k) \otimes J(\mu_l, j_l)).$$

Thus, the following theorem contains the necessary information.

Theorem 2.2 ([1, 3, 6–8]). *Let n and m be positive integers.*
(i) *Let $m \le n$. The matrix $J(0, m) \otimes J(0, n) = N_m \otimes N_n$ has Jordan form*

$$\bigoplus_{i=1}^{m-1} (J(0, i) \oplus J(0, i)) \oplus (J(0, m) \otimes I_{n-m+1}).$$

(ii) *For $\mu \ne 0$, $J(0, m) \otimes J(\mu, n)$ has Jordan form $\bigoplus_{i=1}^n J(0, m)$. Similarly, for $\lambda \ne 0$, $J(\lambda, m) \otimes J(0, n)$ has Jordan form $\bigoplus_{i=1}^m J(0, n)$.*
(iii) *Let $m \le n$. For $\lambda, \mu \ne 0$, $J(\lambda, m) \otimes J(\mu, n)$ has Jordan form*

$$\bigoplus_{j=0}^{j=m-1} (J(\lambda\mu, n - m + 1 + 2j)).$$

Proofs of Theorem 2.2 have appeared in [1, 3, 6–8]. The authors rediscovered this theorem recently and present a new proof of it in §4.

3. THE MAIN THEOREM

Let n_λ^A, m_λ^A, S_A, and T_A be defined as in Definition 2.1. We present necessary and sufficient conditions for the reflexivity of $A \otimes B$.

Definition 3.1. Let T be a linear operator on a finite-dimensional Hilbert space H. We say that A is of *type p* if A is similar to a matrix of the form $J(\lambda, 1) \oplus A_0$ where $\lambda \ne 0$ and $\sigma(A_0) = \{0\}$.

Theorem 3.2. *Let A and B be linear operators on finite-dimensional complex Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, where $\dim \mathscr{H}_1$ and $\dim \mathscr{H}_2$ are both greater than 1. Suppose that neither A and B is of 'type p'. Then $A \otimes B$ is not reflexive if and only if*

(1) $\exists_{\lambda_0 \in S_A \cup T_A, \ \mu_0 \in S_B \cup T_B}$ *such that* $n_{\lambda_0}^A + n_{\mu_0}^B - 1 > \max\{M_{\lambda_0, \mu_0}^{A, B}, 1\}$

where $M_{\lambda_0, \mu_0}^{A, B} = \{n_\lambda^A + n_\mu^B : \lambda \in \sigma(A), \ \mu \in \sigma(B), \ \lambda\mu = \lambda_0\mu_0, \ (\lambda, \mu) \ne (\lambda_0, \mu_0)\}$.

Proof. First, suppose that (1) holds. Let λ_0 and μ_0 be such that $n_{\lambda_0}^A + n_{\mu_0}^B - 1 > \max\{M_{\lambda_0, \mu_0}^{A, B}, 1\}$, and set $\gamma = \lambda_0\mu_0$. We note that $\gamma \ne 0$. Since (1) holds, Theorem 2.1(iii) implies that

$$n_\gamma^{A \otimes B} = n_{\lambda_0}^A + n_{\mu_0}^B - 1$$

and that

$$m_\gamma^{A\otimes B} = n_{\lambda_0}^A + n_{\mu_0}^B - 3.$$

Thus γ does not satisfy the Deddens-Fillmore condition for $A \otimes B$, and $A \otimes B$ is not reflexive.

Next, suppose that (1) does not hold and that γ does not satisfy the Deddens-Fillmore condition for $A \otimes B$. If $\gamma \neq 0$, then $\gamma = \lambda\mu$ where

$$n_\gamma^{A\otimes B} = n_\lambda^A + n_\mu^B - 1.$$

Since (1) does not hold, we may assume without loss of generality that $\lambda \in \sigma(A) \backslash (S_A \cup T_A)$. Then λ satisfies the Deddens-Fillmore condition for A, and $m_\lambda^A > 0$. But, in this case

$$m_\gamma^{A\otimes B} \geq m_\lambda^A + n_\mu^B - 1,$$

and so $n_\lambda^A - m_\lambda^A \geq n_\gamma^{A\otimes B} - m_\gamma^{A\otimes B} > 1$, contradicting the fact that λ satisfies the Deddens-Fillmore condition. Thus every nonzero eigenvalue of $A \otimes B$ satisfies the Deddens-Fillmore condition.

So, suppose that $0 \in \sigma(A \otimes B)$. There are three possible cases:

   (i) $0 \in \sigma(A)$, $0 \notin \sigma(B)$;
   (ii) $0 \in \sigma(B)$, $0 \notin \sigma(A)$;
   (iii) $0 \in \sigma(A) \cap \sigma(B)$.

The first two cases are clearly equivalent.

In case (i) we have

(2) $$n_0^{A\otimes B} = m_0^{A\otimes B} = n_0^A.$$

To see this, let $M = \max_{\mu \in \sigma(B)} n_\mu^B$. If $M > 1$, (2) follows directly from Theorem 2.2(ii). If $M = 1$, then, since $\dim \mathscr{H}_2 > 1$, B has at least two eigenvalues, and so (2) once again follows from Theorem 2.2(ii). Thus, in cases (i) and (ii), 0 satisfies the Deddens-Fillmore condition.

So, suppose we are in case (iii) and that $\sigma(A) = \sigma(B) = \{0\}$. Then, by Theorem 2.2(i), $n_0^{A\otimes B} = N = \min\{n_0^A, n_0^B\}$ and, as above, we may assume $N > 1$. In this case, if $n_0^A \neq n_0^B$, $m_0^{A\otimes B} = N$, and if $n_0^A = n_0^B$, $m_0^{A\otimes B} = N - 1$. Now, suppose $\sigma(A) \neq \{0\}$. Then, if $\sigma(B) = \{0\}$, then $n_0^{A\otimes B} = m_0^{A\otimes B} = n_0^B$, using the fact that A is not of *type p*. If $\sigma(B) \neq \{0\}$, then, using the fact that neither A nor B is of *type p*, with Theorem 2.2(i), (ii), we see that $n_0^{A\otimes B} = m_0^{A\otimes B} = \max\{n_0^A, n_0^B\}$. In each case, 0 satisfies the Deddens-Fillmore condition, and so $A \otimes B$ is reflexive.

The condition for matrices of *type p* to be reflexive are slightly different:
(I) If A and B are both of *type p*, then $A \otimes B$ is reflexive if and only if

$$n_0^B \leq \max\{m_0^B, n_0^A\} + 1 \quad \text{and} \quad n_0^A \leq \max\{m_0^A, n_0^B\} + 1.$$

(II) If one of the two matrices, say A, is of *type p* and B is not, then $A \otimes B$ is reflexive if and only if all nonzero λ in the spectrum of B satisfy the Deddens-Fillmore condition and $n_0^B \leq \max\{m_0^B, n_0^A\} + 1$.

We present some corollaries detailing the more interesting consequences.

Corollary 3.3. *Let A and B be linear operators on finite-dimensional complex Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, where $\dim\mathscr{H}_1$ and $\dim\mathscr{H}_2$ are both greater than 1.*

(i) *If $S_A \cup T_A = \varnothing \Rightarrow A \otimes B$ is reflexive for all B not of 'type p'.*
(ii) *If A and B are both reflexive, then $A \otimes B$ is reflexive.*
(iii) *If A is reflexive and $S_A = \varnothing$, then $A \otimes B$ is reflexive for all B.*
(iv) *If $\sigma(A) = \{0\}$, then $A \otimes B$ is reflexive for all B not of 'type p'.*

Corollary 3.4. *Let T be a finite rank operator on a Banach space E. Let $E_1 \oplus E_2$ be a direct sum decomposition of E such that E_1 is finite dimensional, $E_1 \in \operatorname{Lat} T$, and $E_2 \subseteq \ker T$. Then T is a reflexive operator if and only if $T_{|E_1}$ is a reflexive operator. Thus (i)–(iv) of Corollary 3.3 hold for finite rank operators A and B.*

Proof. Suppose that $T_{|E_1}$ is a reflexive operator. Let $S \in \operatorname{Alg}(\operatorname{Lat}(T))$. Then $S_{|E_2} \in \operatorname{Alg}(\operatorname{Lat}(0_{E_2}))$, so $S_{|E_2} = \lambda I_{E_2}$. Let $v \in E_2$, and set $E_1' = \operatorname{span}(E_1, \{v\})$ and $E_2' =$ the complement in E of E_1'. Since $T_{|E_1}$ is reflexive, the Deddens-Fillmore theorem implies that $T_{|E_1'}$ is reflexive. Thus $S_{|E_1'} = q(T)$, and $S_{|E_2'} = \lambda_{E_2'}$ with $q(0) = \lambda$. So $S = q(T)$, $\operatorname{Alg}(\operatorname{Lat}(T))$ is composed entirely of polynomials in T, and T is reflexive. On the other hand, if T is reflexive, the fact that $\operatorname{Lat} T = \operatorname{Lat} T_{|E_1} \oplus \operatorname{Lat} 0_{E_2}$ implies immediately that $T_{|E_1}$ is reflexive.

4. Proof of Theorem 2.2

It is well known and easy to derive that, if A is any linear operator on finite-dimensional Hilbert space, then

$$\dim(\ker(A - \lambda)^k) - \dim(\ker(A - \lambda)^{k-1})$$

(3)
$$= \text{number of Jordan blocks belonging to } \lambda \text{ of size } \geq k$$
$$\text{in the Jordan decomposition of } A.$$

So, let $T_{m,n} = J(\lambda, m) \otimes J(\mu, n) - \lambda\mu I_{mn}$. For $\lambda = \mu = 0$, $\operatorname{rank}(T_{m,n}^k) = (m-k)(n-k)$; for $\lambda = 0$ and $\mu \neq 0$, $\operatorname{rank}(T_{m,n}^k) = (m-k)n$; while for $\lambda \neq 0$ and $\mu = 0$, $\operatorname{rank}(T_{m,n}^k) = (n-k)m$ (easily calculated by multiplying the block matrices and noticing that the rank of the resulting matrix is exactly equal to the number of nonzero rows). (i) and (ii) follow directly.

The proof of (iii) is more complicated. First, note that the matrix of $T_{m,n}$ with respect to an appropriate basis is of the form

$$T_{m,n} = \begin{pmatrix} \lambda N_n & J(\mu, n) & 0 & \cdots & 0 \\ 0 & \lambda N_n & J(\mu, n) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \lambda N_n & J(\mu, n) \\ 0 & \cdots & \cdots & 0 & \lambda N_n \end{pmatrix}.$$

Let

$$X = \begin{pmatrix} (J(\mu, n))^{m-1} & 0 & \cdots & \cdots & 0 \\ 0 & (J(\mu, n))^{m-2} & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & (J(\mu, n)) & 0 \\ 0 & \cdots & \cdots & 0 & I_n \end{pmatrix}.$$

It is easy to see that X is invertible and that $C_{m,n} = X^{-1}(T_{m,n})X$ is of the form

$$C_{m,n} = \begin{pmatrix} \lambda N_n & I_n & 0 & \cdots & 0 \\ 0 & \lambda N_n & I_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \lambda N_n & I_n \\ 0 & \cdots & \cdots & 0 & \lambda N_n \end{pmatrix}.$$

Notice that $\operatorname{rank}(T_{m,n})^k = \operatorname{rank}(C_{m,n})^k$ and that

$$(C_{m,n})^k = \begin{pmatrix} a_0 N_n^k & a_1 N_n^{k-1} & \cdots & a_{m-1} N_n^{k-m+1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & a_0 N_n^k & a_1 N_n^{k-1} \\ 0 & \cdots & 0 & a_0 N_n^k \end{pmatrix}$$

where $a_i = \lambda^{k-i}\binom{k}{i}$ and we define $N_n^{-j} = 0$ for $j > 0$ and $\binom{k}{i} = 0$ if $i > k$.

We use two rather technical lemmas to determine $\operatorname{rank}(C_{n,m})^k$.

Lemma 4.1. *Let* $p, k, r \in \mathscr{Z}$, $k \geq p \geq 0$, $p + r \geq 0$. *Let* A *be the matrix with coordinates*

$$(4) \qquad a_{ij} = \binom{k}{p - i + j} \qquad (i, j = 0, 1, \dots, p + r)$$

where $\binom{k}{l}$ *is defined to be zero for* $l > k$ *or* $l < 0$. *Then*

$$\det A = \frac{\binom{k}{p}\binom{k+1}{p+1}\cdots\binom{k+p+r}{2p+r}}{\binom{k+r}{p+r}\binom{k+r-1}{p+r-1}\cdots\binom{k-p}{0}}.$$

Proof. Notice that, for notational convenience, our matrix has a '0th' row and a '0th' column.

Let $A_{k,p,r}$ be the matrix defined in (4). We claim that

$$\det(A_{k,p,r}) = \frac{\binom{k}{p}}{\binom{k+r}{p+r}} \det A_{k+1,p+1,r-2}$$

which proves the lemma (by induction on $p + r$).

Given a matrix $A_{k,p,r}$ to see that the claim is true, just subtract $(\frac{a_{i,0}}{a_{i-1,0}} = \frac{p-i+1}{k-p+i})$ times the $(i - 1)$st row from the ith row, and put the result into the ith row. Since

$$\binom{k}{p - i + j} - \frac{p - i + 1}{k - p + i}\binom{k}{p - i + j + 1} = \frac{j}{k - p + i}\binom{k+1}{p - i + j + 1},$$

the new matrix has entries

$$b_{ij} = \begin{cases} a_{i,j}, & i = 0, 0 \le j \le p + r, \\ 0, & 1 \le i \le p + r, j = 0, \\ \dfrac{j}{k - p + i} \dbinom{k+1}{p - i + j + 1}, & 1 \le i, j \le p + r, \end{cases}$$

so that

$$\det A_{k,p,r} = \binom{k}{p} \det[b_{ij}]_{i,j=1}^{p+r} = \frac{\binom{k}{p}}{\binom{k+r}{p+r}} \det A_{k+1,p+1,r-2}.$$

Lemma 4.2. *Let* $m, n, k \in \mathcal{N}$, $m \le n$. *Then*

$$\mathrm{Rank}(T_{m,n})^k = \begin{cases} m(n-k) & \text{if } k \le m - n, \\[2mm] \dfrac{(m+n-k)^2}{4} & \text{if } n + m - 1 > k > n - m \\[2mm] & \text{and } m + n - k \text{ even}, \\[2mm] \dfrac{(m+n-k+1)(m+n-k-1)}{4} \\[2mm] & \text{if } n + m - 1 > k > n - m \\[2mm] & \text{and } m + n - k \text{ odd}, \\[2mm] 0 & \text{if } k \ge m + n - 1. \end{cases}$$

Proof. We divide the matrix $(C_{m,n})^k$ into m blocks of n rows and denote by $\mathrm{ROW}(I, J)$ the Jth row of the Ith block. The rows can then be divided into $m + n - 2$ equivalence classes $\{R_p\}_{p=2}^{m+n}$ where

$$R_p = \{R(I, J): I + J = p\}.$$

We note that R_p has α_p elements where

$$\alpha_j = \begin{cases} j - 1, & j = 2, \dots, m, \\ m, & j = m + 1, \dots, n + 1, \\ m + n + 1 - j, & j = n + 1, \dots, m + n, \end{cases}$$

and that the nonzero rows of $(C_{m,n})^k$ are precisely those rows in the equivalence classes $R_2, R_3, \dots, R_{m+n-k}$.

Similarly, by dividing $(C_{m,n})^k$ into m blocks of n columns we define $\mathrm{COL}(I, J)$ and the equivalence classes C_p $(p = 2, 3, \dots, m+n)$. (As before C_p has α_p elements.) It is easily seen that the nonzero columns of the elements of the class R_j are precisely the elements of the class C_{j+k}. Thus $\{R_p\}_{p=2}^{m+n}$ is a pairwise orthogonal set and

$$\mathrm{rank}((C_{m,n})^k) = \sum_{j=2}^{m+n-k} \dim(\mathrm{span}(R_j))I.$$

Now, let

$$p_j = \min(\alpha_j, \alpha_{j+k})$$
$$= \min(\text{number of elements of } R_j, \text{number of elements of } C_{j+k}).$$

Since the matrix with rows equal to the elements of R_j contains a submatrix of form (4) with $p + r = p_j$, we get

$$\text{Rank}((C_{m,n})^k) = \sum_{j=2}^{m+n-k} p_j.$$

It is easily seen that

$$p_j = \begin{cases} \alpha_j, & 2 \le j \le \gamma, \\ \alpha_{j+k}, & \gamma < j \le m+n-k, \end{cases}$$

where

$$\gamma = \begin{cases} n+1-k & \text{for } k \le n-m, \\ \dfrac{m+n-k}{2}+1 & \text{for } k > n-m \text{ and } m+n-k \text{ even}, \\ \dfrac{m+n-k+1}{2} & \text{for } k > n-m \text{ and } m+n-k \text{ odd}, \end{cases}$$

and now a simple computation finishes the proof.

We will now prove Theorem 2.2(iii). Using Lemma 4.2 we see that, if we set $\kappa_k = \dim(\ker(T_{m,n})^k) - \dim(\ker(T_{m,n})^{k-1})$, then

$$\kappa_k = \begin{cases} m & \text{if } 1 \le k \le n-m+1, \\ m + \left[\dfrac{k-(n-m)}{2} \right] & \text{if } n-m+2 \le k \le m+n-1, \\ 0 & \text{if } k > n+m-1, \end{cases}$$

where $[r] = $ the greatest integer less than r for each real number r. Thus, using equation (3),

$$\text{\# of blocks of size } k = \begin{cases} 0 & \text{if } k < n-m+1, \\ 1 & \text{if } n-m+1 \le k \le n+m-1 \\ & \quad \text{and } k-(n-m) \text{ odd}, \\ 0 & \text{if } n-m+1 \le k \le n+m-1 \\ & \quad \text{and } k-(n-m) \text{ even}, \\ 0 & \text{if } k > n+m-1, \end{cases}$$

which finishes the proof.

ACKNOWLEDGMENT

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