

QUADRATIC DESCENT OF INVOLUTIONS IN DEGREE 2 AND 4

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ABSTRACT. If K/F is a quadratic extension, we give necessary and sufficient conditions in terms of the discriminant (resp. the Clifford algebra) for a quadratic form of dimension 2 (resp. 4) over K to be similar to a form over F . We give similar criteria for an orthogonal involution over a central simple algebra A of degree 2 (resp. 4) over K to be such that $A = A' \otimes_F K$, where A' is invariant under the involution. This leads us to an example of a quadratic form over K which is not similar to a form over F but such that the corresponding involution comes from an involution defined over F .

INTRODUCTION

For a quadratic extension K of a field F of characteristic different from 2, we consider the two following questions:

- (1) For a quadratic form q defined over K , when is it similar to a form $q' \otimes_F K$, where q' is defined over F ?
- (2) Let A be a central simple algebra over K endowed with an involution σ of the first kind and orthogonal type. When is σ equal to $\sigma' \otimes_F K$, where $\sigma' = \sigma|_{A'}$ for some σ -invariant F -subalgebra A' of A ?

In the first section we recall some facts about the discriminant and the Clifford algebra of an involution. Next, we give necessary and sufficient conditions in terms of discriminants to have “quadratic descent” for quadratic forms of dimension 2 and involutions in degree 2. In the third section, we give criteria for 4-dimensional quadratic forms and for orthogonal involutions in degree 4 using Clifford algebras. The last section is devoted to the comparison between quadratic forms and involutions. We construct an example of a 4-dimensional quadratic form over K which is not similar to a form over F but such that the adjoint involution with respect to this form comes from an involution defined over F .

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1. PRELIMINARIES ON INVOLUTIONS

Let A be a central simple algebra of degree n over a field F . An involution on A is of the first kind if it leaves F elementwise invariant. There are two types of involutions of the first kind on A : the F -subspace of σ -invariant elements of A has dimension $\frac{n(n+1)}{2}$ or $\frac{n(n-1)}{2}$ and σ is said of *orthogonal type* in the first case and of *symplectic type* in the second case [S, Definition 8.7.6]. Observe that in the special case where $A = \text{End}_F V$ for some F -vector space V , orthogonal involutions on A are in one-to-one correspondence with similarity classes of quadratic forms on V . Indeed, any involution of the first kind on A is the adjoint involution with respect to some F -bilinear form b on V , i.e.,

$$b(fx, y) = b(x, \sigma(f)(y)) \quad \text{for } x, y \in V \text{ and } f \in A$$

[S, p. 302]. This bilinear form b is determined up to scalars by σ , and b is symmetric if σ has orthogonal type and symplectic if σ has symplectic type. Hence, an orthogonal involution on A corresponds to the similarity class of the quadratic form

$$q(x) = b(x, x);$$

we denote it by $\sigma = I_q$. For an involution σ of the first kind on a central simple algebra A of even degree over F , Knus, Parimala, and Sridharan [KPS] have defined the *discriminant*

$$\text{disc } \sigma = \text{Nrd } a \cdot F^{\times 2}$$

for any $a \in A^\times$ such that $\sigma(a) + a = 0$ if σ is orthogonal, and $\sigma(a) - a = 0$ if σ is symplectic. In particular, if $A = \text{End}_F V$ and $\sigma = I_q$, then $\text{disc } \sigma = \text{disc } q = \det q \cdot F^{\times 2}$.

If σ is orthogonal, there is another invariant attached to it: its *Clifford algebra* $C(A, \sigma)$. For precise definitions we refer to the papers of Tits [Ti] or Jacobson [J]. We recall here some facts we will need.

Suppose that A has degree $2m$ over F , that σ has orthogonal type, and that $\text{disc } \sigma = d \cdot F^{\times 2}$ for some $d \in F^\times$.

(1.1) If $d \notin F^{\times 2}$, then $C(A, \sigma)$ is a central simple algebra of degree 2^{m-1} over $F(\sqrt{d})$.

(1.2) If $d \in F^{\times 2}$, then $C(A, \sigma)$ is a direct sum of two central simple algebras of degree 2^{m-1} over F [Ti, Corollaire 2].

(1.3) If $A = \text{End}_F V$ and $\sigma = I_q$, then $C(A, \sigma) = C_0(q)$ [Ti, Théorème 2].

In the special case where $\deg A = 4$, we have:

(1.4) If $\text{disc } \sigma = 1 \cdot F^{\times 2}$, then $A = Q_1 \otimes_F Q_2$, where $\sigma|_{Q_i}$ is the standard involution on the quaternion algebra Q_i and $C(A, \sigma) = Q_1 \times Q_2$ [Ti, Proposition 7].

(1.5) If $\text{disc } \sigma = d \cdot F^{\times 2}$, where $d \in F^\times \setminus F^{\times 2}$, then $\text{cor}_{F(\sqrt{d})/F}(C(A, \sigma)) = A$ [T, Theorem 4.3]. Moreover the involution σ on A induces an involution σ^* on $C(A, \sigma)$ which is in this case the standard involution of the $F(\sqrt{d})$ -quaternion algebra $C(A, \sigma)$ and $\text{cor}_{F(\sqrt{d})/F}(\sigma^*) = \sigma$ (see [KMRT]).

2. DIMENSION 2 AND DEGREE 2

First recall that for a quadratic extension K of F , the norm map $N_{K/F}: K \rightarrow F$ induces an exact sequence

$$(2.1) \quad F^\times / F^{\times 2} \rightarrow K^\times / K^{\times 2} \xrightarrow{N_{K/F}} F / F^{\times 2}.$$

We consider first the case of quadratic forms of dimension 2.

2.2. Proposition. *For a nondegenerate quadratic form q of dimension 2 over K , q is similar to $q' \otimes_F K$ for some quadratic form q' over F if and only if $N_{K/F}(\text{disc } q) = 1 \cdot F^{\times 2}$.*

Proof. Suppose that $q = \langle a, b \rangle = \langle k \rangle \langle \alpha, \beta \rangle$ for $a, b, k \in K$ and $\alpha, \beta \in F$. Then $\text{disc } q = ab \cdot K^{\times 2} = \alpha\beta \cdot K^{\times 2} \in F^\times \cdot K^{\times 2}$, and therefore $N_{K/F}(\text{disc } q) = 1 \cdot F^{\times 2}$. Conversely, let $q = \langle a, b \rangle$, and suppose that $N_{K/F}(ab \cdot K^{\times 2}) = 1 \cdot F^{\times 2}$. Then by exactness of sequence (2.1), we have $ab \cdot K^{\times 2} = f \cdot K^{\times 2}$ for some $f \in F^\times$. Hence $q = \langle a, b \rangle = \langle ab \rangle \langle 1, ab \rangle = \langle ab \rangle \langle 1, f \rangle$. \square

In the proof of the corresponding result for involutions over central simple algebras of degree 2, we need some properties of the corestriction of quaternion algebras.

2.3. Lemma. *If $A = (a, b)_K$ with $a \in F^\times$ and $b \in K^\times$ is such that $\text{cor}_{K/F} A$ splits, there exists some $c \in F^\times$ such that $A = (a, c)_F \otimes_F K$.*

Proof. If $a \in K^{\times 2}$, we can take $c = 1$ since $A = M_2(K) = (a, 1)_F \otimes_F K$. If $a \notin K^{\times 2}$, then $K(\sqrt{a})$ is a field and let σ_1, σ_2 be the F -automorphisms of $K(\sqrt{a})$ leaving K and $F(\sqrt{a})$ respectively invariant. By the projection formula for the corestriction, we have that

$$\text{cor}_{K/F}(a, b)_K \text{ is similar to } (a, N_{K/F}(b))_F.$$

Hence $(a, N_{K/F}(b))_F$ splits, and this implies that $N_{K/F}(b)$ is a norm in $F(\sqrt{a})$:

$$N_{K/F}(b) = b\sigma_2(b) = u\sigma_1(u) \text{ for some } u \in F(\sqrt{a}).$$

Now, observe that

if $1 + ub^{-1} = 0$, then $b = -u \in K \cap F(\sqrt{a}) = F$ and we can take $b = c$;
if $1 + ub^{-1} \neq 0$, then

$$\begin{aligned} N_{K(\sqrt{a})/K}(1 + ub^{-1}) &= (1 + ub^{-1})\sigma_1(1 + ub^{-1}) \\ &= b^{-1}(b + u + \sigma_1(u) + \sigma_2(b)) \\ &= b^{-1} \cdot c \text{ with } c \in F. \end{aligned}$$

Hence $b = c \cdot N_{K(\sqrt{a})/K}(1 + ub^{-1})$ and $(a, b)_K = (a, c)_K = (a, c)_F \otimes_F K$. \square

We will also use the fact that, by a theorem of Albert [A, Theorem X.21], a quaternion algebra A over K is isomorphic to $A' \otimes_F K$ for some F -quaternion algebra A' if and only if A has an involution of the second kind or, equivalently, by Albert-Riehm-Scharlau's theorem [S, Theorem 8.9.5], if and only if $\text{cor}_{K/F} A$ splits.

2.4. Proposition. *Let A be a quaternion algebra over K and σ an involution of the first kind on A . Then $A = A' \otimes_F K$, where A' is a σ -invariant F -subalgebra of A if and only if $N_{K/F}(\text{disc } \sigma) = 1 \cdot F^{\times 2}$ and $\text{cor}_{K/F} A$ splits.*

Proof. If $A = A' \otimes_F K$ with $\sigma(A') = A'$, we have that $\text{cor}_{K/F} A$ splits by the remark preceding the proposition. If σ has symplectic type, then $\text{disc } \sigma = 1 \cdot K^{\times 2}$ and $N_{K/F}(\text{disc } \sigma) = 1 \cdot F^{\times 2}$. If σ has orthogonal type, take $a \in A'^{\times}$ such that $\sigma(a) = -a$. Then $\text{disc } \sigma = \text{Nrd } a \cdot K^{\times 2} \in F^{\times} \cdot K^{\times 2}$. Hence $N_{K/F}(\text{disc } \sigma) = 1 \cdot F^{\times 2}$.

To prove the converse, let $\sigma = \text{Int } x \circ (-)$, where $(-)$ is the standard involution on A and $x \in A^{\times}$. Then $\text{disc } \sigma = \text{Nrd } x \cdot K^{\times 2} = -x^2 \cdot K^{\times 2}$. Since $N_{K/F}(\text{disc } \sigma) = N_{K/F}(-x^2 \cdot K^{\times 2})$ is trivial in $F^{\times}/F^{\times 2}$, by exactness of sequence (2.1), we can assume that $-x^2 \in F^{\times}$. Furthermore, $A = (x^2, b)_K$ for some $b \in K^{\times}$, and since $\text{cor}_{K/F} A$ splits, by Lemma 2.3 there exists $c \in F^{\times}$ such that $A = (x^2, c)_F \otimes_F K$. Hence there is some $y \in A$ such that $y^2 = c$ and $xy = -xy$. We take for A' the subalgebra of A generated by x and y ; it is stable by $\sigma = \text{Int } x \circ (-)$ since

$$\sigma(x) = x\bar{x}x^{-1} = -x \quad \text{and} \quad \sigma(y) = x\bar{y}x^{-1} = -xyx^{-1} = y. \quad \square$$

3. DIMENSION 4 AND DEGREE 4

The proof of the result for quadratic forms of dimension 4 uses the following theorem of Wadsworth.

3.1. Proposition [W]. *Let q, q' be two quadratic forms of dimension 4 over F such that $\text{disc } q = \text{disc } q' = d \cdot F^{\times 2}$. Then q and q' are similar over F if and only if they are similar over $F(\sqrt{d})$.*

Here is a short proof using the properties of the Clifford algebra of an involution over a central simple algebra of degree 4.

Proof. One way of the statement is trivial. To prove the other one, observe that if

$$q \otimes F(\sqrt{d}) \sim q' \otimes F(\sqrt{d}),$$

then

$$C_0(q) \otimes F(\sqrt{d}) \cong C_0(q') \otimes F(\sqrt{d})$$

since similar quadratic forms have isomorphic even Clifford algebras. Since we consider q and q' up to similarity, we can write $q = \langle 1, -a, -b, abd \rangle$ and $q' = \langle 1, -a', -b', a'b'd \rangle$ for $a, b, a', b' \in F$. Hence

$$C_0(q) = (a, b)_F \otimes_F F(\sqrt{d}) \quad \text{and} \quad C_0(q') = (a', b')_F \otimes_F F(\sqrt{d}).$$

So, $C_0(q) \otimes F(\sqrt{d}) \cong C_0(q') \otimes F(\sqrt{d})$ implies that $C_0(q) \cong C_0(q')$.

Denoting by γ the standard involution on $C_0(q)$ we have that

$$\text{cor}_{F(\sqrt{d})/F}(C_0(q), \gamma) = (M_4(F), I_q)$$

and

$$\text{cor}_{F(\sqrt{d})/F}(C_0(q'), \gamma) = (M_4(F), I_{q'})$$

by (1.5). Hence

$$(M_4(F), I_q) \cong (M_4(F), I_{q'}),$$

and this means that q is similar to q' . \square

3.2. Proposition. *Let q be a 4-dimensional quadratic form over K . Then q is similar to $q' \otimes_F K$ for q' a 4-dimensional quadratic form over F if and only if there exists some $d \in F^\times$ such that $\text{disc } q = d \cdot K^{\times 2}$ and $C_0(q) = Q \otimes_F K(\sqrt{d})$, where Q is a quaternion algebra over F .*

Proof. Since we consider q up to similarity we may write $q = \langle 1, -a, -b, abd \rangle$, where $d \cdot K^{\times 2} = \text{disc } q$, and therefore $C_0(q) = (a, b)_K \otimes_K K(\sqrt{d})$.

If q is similar to the form $\langle 1, -\alpha, -\beta, \gamma \rangle$ with $\alpha, \beta, \gamma \in F^\times$, then $\text{disc } q = \alpha\beta\gamma \cdot K^{\times 2} = d \cdot K^{\times 2}$ and we can suppose that $d \in F$. Moreover, since similar quadratic forms have isomorphic even Clifford algebras, we have

$$\begin{aligned} C_0(q) &\cong C_0(\langle 1, -\alpha, -\beta, \alpha\beta d \rangle \otimes_F K) \\ &\cong (\alpha, \beta)_F \otimes_F F(\sqrt{d}) \otimes_F K = (\alpha, \beta)_F \otimes_F K(\sqrt{d}). \end{aligned}$$

Conversely, suppose that $C_0(q) = (a, b)_K \otimes_K K(\sqrt{d})$ is isomorphic to $(\alpha, \beta)_F \otimes_F K(\sqrt{d})$ for some $\alpha, \beta \in F$. If $d \in K^{\times 2}$, then $C_0(q) = (a, b)_K \times (a, b)_K = (\alpha, \beta)_F \otimes K \times (\alpha, \beta)_F \otimes K$, and so $(a, b)_K = (\alpha, \beta)_F \otimes K$. The norm forms of these quaternion algebras are isometric, and

$$q = \langle 1, -a, -b, ab \rangle = \langle 1, -\alpha, -\beta, \alpha\beta \rangle \otimes_F K.$$

If $d \notin K^{\times 2}$, then $(a, b)_K \otimes K(\sqrt{d}) = (\alpha, \beta)_F \otimes K(\sqrt{d})$ and the norm forms are isometric:

$$\begin{aligned} \langle 1, -a, -b, abd \rangle \otimes_K K(\sqrt{d}) &= \langle 1, -\alpha, -\beta, \alpha\beta \rangle \otimes_F F \otimes_K K(\sqrt{d}) \\ &= \langle 1, -\alpha, -\beta, \alpha\beta d \rangle \otimes_F K \otimes_K K(\sqrt{d}). \end{aligned}$$

By Proposition 3.1, the forms $q = \langle 1, -a, -b, abd \rangle$ and $\langle 1, -\alpha, -\beta, \alpha\beta d \rangle \otimes_F K$ must be similar. \square

Here is the corresponding result for orthogonal involutions on central simple algebras of degree 4.

3.3. Theorem. *Let A be a central simple algebra of degree 4 over K endowed with an involution σ of type $+1$. Then $A = A' \otimes_F K$, where A' is a σ -invariant F -subalgebra of A if and only if there exists some $d \in F^\times$ such that $\text{disc } \sigma = d \cdot K^{\times 2}$ and*

- if $d \notin F^{\times 2}$: $C(A, \sigma) = C \otimes_{F(\sqrt{d})} K(\sqrt{d})$ for some $F(\sqrt{d})$ -quaternion algebra C ;
- if $d \in F^{\times 2}$: $C(A, \sigma) = C_1 \times C_2$, where $C_i = A_i \otimes_F K$ for some F -quaternion algebras A_i .

Proof. If $A = A' \otimes_F K$, then $C(A, \sigma) = C(A', \sigma') \otimes K$, where $\sigma' = \sigma|_{A'}$ and $\text{disc } \sigma = \text{disc } \sigma' \cdot K^{\times 2} = d \cdot K^{\times 2}$ with $d \in F^\times$. If $d \in F^{\times 2}$, then $C(A', \sigma') = A_1 \times A_2$, where A_1 and A_2 are quaternion algebras over F . Hence $C(A, \sigma) = A_1 \otimes K \times A_2 \otimes K$. If $d \notin F^{\times 2}$, then $C(A', \sigma')$ is a quaternion algebra over $F(\sqrt{d})$, and so $C(A, \sigma) = C(A', \sigma') \otimes_F K = C(A', \sigma') \otimes_{F(\sqrt{d})} K(\sqrt{d})$.

Conversely, suppose first that $\text{disc } \sigma = d \cdot K^{\times 2}$ with $d \in F^{\times 2}$ and $C(A, \sigma) = A_1 \otimes K \times A_2 \otimes K$. Since $\text{disc } \sigma = 1 \cdot K^{\times 2}$, we have by (1.4) that $A = Q_1 \otimes_K Q_2$, where $\sigma|_{Q_i}$ is the standard involution on Q_i and $C(A, \sigma) = Q_1 \times Q_2$. Hence, $Q_1 = A_1 \otimes K$ and $Q_2 = A_2 \otimes K$. Therefore, $A = A_1 \otimes_F A_2 \otimes_F K$, where $A_1 \otimes_F A_2$

is a σ -invariant subalgebra since $\sigma|_{A_i}$ is the standard involution. In the case where $\text{disc } \sigma = d \cdot K^{\times 2}$ with $d \in F^\times \setminus F^{\times 2}$ and $C(A, \sigma) = C \otimes_{F(\sqrt{d})} K(\sqrt{d})$, by property (1.5) we have

$$A = \text{cor}_{K(\sqrt{d})/K} C(A, \sigma) = \text{cor}_{F(\sqrt{d})/F}(C) \otimes K.$$

It can be checked that $\text{cor}_{F(\sqrt{d})/F}(C)$ is a σ -invariant subalgebra of A . This follows from the fact that σ^* is the standard involution on the $K(\sqrt{d})$ -quaternion algebra $C(A, \sigma)$ and that $\text{cor}_{K(\sqrt{d})/K}(\sigma^*) = \sigma$. \square

4. COMPARISON BETWEEN QUADRATIC FORMS AND INVOLUTIONS

In this section, we suppose that $A = \text{End}_K V$ for some vector space V of dimension 4 over K and that $\sigma = I_q$ for some quadratic form q on V . Clearly, q is similar to $q' \otimes_F K$ for some quadratic form q' over F if and only if $A = A' \otimes_F K$, where A' is a σ -invariant subalgebra of A isomorphic to $M_4(F)$. By Theorem 3.3, this is equivalent to saying that $\text{disc } \sigma = \text{disc } q = d \cdot K^{\times 2}$ for some $d \in F^\times$ and

- if $d \in F^{\times 2}$: $C(A, \sigma) = C_0(q) = A_1 \otimes K \times A_2 \otimes K$, where $A' = A_1 \otimes A_2$ splits;
- if $d \notin F^{\times 2}$: $C(A, \sigma) = C_0(q) = C \otimes_{F(\sqrt{d})} K(\sqrt{d})$, where $A' = \text{cor}_{F(\sqrt{d})/F}(C)$ splits.

The assumption that A' splits implies that $A_1 \cong A_2$ in the first case, that is, $C(A, \sigma) = C_0(q) = (A_1 \otimes K) \times (A_2 \otimes K) = A_1 \otimes_F K(\sqrt{d})$; in the second case, this means that $\text{cor}_{F(\sqrt{d})/F}(C)$ splits. Hence $C = C' \otimes_F F(\sqrt{d})$, and therefore $C_0(q) = C' \otimes_F K(\sqrt{d})$. So this yields another proof of one way of Proposition 3.2. Moreover, we see that if q is similar to $q' \otimes_F K$, then $I_q = I_{q'} \otimes_F K$.

The converse is false; this is a consequence of the following.

4.1. Proposition. *There exists a biquadratic extension M/F and a quaternion algebra Q over M such that for any quadratic subfield N of M we have $Q = Q' \otimes_N M$ for some N -quaternion algebra Q' , but there is no F -quaternion algebra Q'' such that $Q = Q'' \otimes_F M$.*

Proof. Let D be a division algebra of degree 8 and exponent 2 which is not a tensor product of quaternion algebras [ART]. The algebra D has a maximal subfield L such that $\text{Gal}(L/F) \cong (\mathbb{Z}/2\mathbb{Z})^3$ [R1]. Let $L = K_1 \otimes_F K_2 \otimes_F K_3$, where K_i/F are quadratic extensions. Choose $M = K_1 \otimes_F K_2$ and $Q = C_D M$. Then Q has no F -quaternion subalgebra since D is indecomposable. Nevertheless, for any quadratic subfield N of M there is a N -quaternion algebra Q' such that $Q = Q' \otimes_N M$. Indeed, for any given F -automorphism of L , there is an involution on D which agrees with it [R2, Theorem 7.2.45]. In particular, $Q = C_D M$ admits involutions of the second kind σ such that $\sigma|_M$ leaves elementwise invariant any given quadratic subfield N of M . By Albert's theorem [A, Theorem X.21], this implies that $Q = C_D M = Q' \otimes_N M$ for some quaternion algebra Q' over N . \square

This allows us to give an example of a quadratic form q which has no descent but such that $\sigma = I_q$ admits a descent. Let $M = F(\sqrt{k}, \sqrt{d})$ and Q a quaternion algebra over M as in Proposition 4.1. Then $Q = (a, b)_{F(\sqrt{k})} \otimes_{F(\sqrt{k})}$

M for some $a, b \in F(\sqrt{k})$, but Q has no F -quaternion subalgebra. Therefore, $q = \langle 1, -a, -b, abd \rangle$ is a form over $F(\sqrt{k})$ which has no descent over F by Proposition 3.2, since $C_0(q) = Q$. On the other hand, $Q = C_0(q)$ admits a descent over $F(\sqrt{d})$, and therefore, by Theorem 3.3, the involution I_q admits a descent over F .

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