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ON POLYNOMIALLY BOUNDED OPERATORS WITH RICH SPECTRUM

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ABSTRACT. D. Westood (J. Funct. Anal. **66** (1986), 96–104) proved that C_{00}-contractions with dominating spectrum are in $A_{\aleph_0}$. We generalize this result to polynomially bounded operators.

1. INTRODUCTION

Let $\mathscr{H}$ be a complex, separable, infinite dimensional Hilbert space, and let $\mathscr{B}(\mathscr{H})$ be the algebra of all bounded, linear operators on $\mathscr{H}$. Recall that an operator $T \in \mathscr{B}(\mathscr{H})$ is called *polynomially bounded* (notation $T \in (PB)(\mathscr{H})$) if there exists a constant $K \geq 1$ such that for every polynomial p,

$$\|p(T)\| \leq K \sup\{|p(z)| : |z| = 1\}. \tag{1}$$

Of course, all contraction operators in $\mathscr{B}(\mathscr{H})$ are polynomially bounded, and in the past fifteen years the theory of dual algebras generated by a single contraction operator has been used very succesfully to obtain information about the structure of such operators (see for example [1], [2], [5], [6]). More recently (cf. [11], [12], [13], [15], etc.), researchers have begun to use the theory of dual algebras generated by an arbitrary polynomially bounded operator to extract structural information about such operators. As was pointed out in [11], however, many parts of the theory for contraction operators do not readily generalize to the case of polynomially bounded operators. The purpose of this note is to make a modest contribution to this theory, by proving a generalization (Theorem 2 below) of the main result in [16] and one of the results in [11]. Before stating Theorem 2, we recall some notation and definitions from this theory.

If T is in $\mathscr{B}(\mathscr{H})$ and $\mathscr{M}$ is a (closed) subspace of $\mathscr{H}$, then $T_{\mathscr{M}}$ denotes the compression of T to $\mathscr{M}$, i.e., $T_{\mathscr{M}} = P_{\mathscr{M}} T_{|\mathscr{M}}$, where $P_{\mathscr{M}}$ denotes the orthogonal projection from $\mathscr{H}$ onto $\mathscr{M}$. Also the spectrum of T, the point spectrum of T and the essential spectrum of T will be denoted by $\sigma(T)$, $\sigma_p(T)$ and $\sigma_e(T)$, respectively. Moreover, $C_{00}(\mathscr{H})$ is the set of all operators T in $\mathscr{B}(\mathscr{H})$ such that the sequences $\{T^n\}_{n=1}^{\infty}$, $\{T^{*n}\}_{n=1}^{\infty}$ converge to 0 in the strong operator topology on $\mathscr{B}(\mathscr{H})$.

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It is well known (cf. [9]) that $\mathcal{B}(\mathcal{H})$ is the dual space of the Banach space $\mathcal{C}_1(\mathcal{H})$ of trace-class operators on $\mathcal{H}$ equipped with the trace-norm $\|\ \|_1$, and the duality is implemented by the bilinear form $\langle T\,,\,L\rangle = \mathrm{trace}(TL)$, $T \in \mathcal{B}(\mathcal{H})$, $L \in \mathcal{C}_1(\mathcal{H})$. If T is an operator in $\mathcal{B}(\mathcal{H})$, $\mathcal{A}_T$ will denote the dual algebra generated by T (i.e., the smallest weak*-closed algebra containing T and the identity operator on $\mathcal{H}$), $\mathcal{Q}_T$ $(= \mathcal{C}_1/{}^{\perp}\mathcal{A}_T)$ the natural predual of $\mathcal{A}_T$. For any $L \in \mathcal{C}_1(\mathcal{H})$ the corresponding element in $\mathcal{Q}_T$ will be denoted by $[L]_T$. In particular, for any vectors x and y in $\mathcal{H}$, $[x \otimes y]_T$ is the image in $\mathcal{Q}_T$ of $x \otimes y$, where $x \otimes y$ denotes the usual rank one operator in $\mathcal{B}(\mathcal{H})$.

As usual $\mathbf{D}$ denotes the open unit disc in $\mathbf{C}$, and $\mathbf{T} = \partial\,\mathbf{D}$. If E is a measurable subset of $\mathbf{T}$ (with respect to normalized Lebesgue measure $\mathbf{m}$ on $\mathbf{T}$), a set $\Lambda \subset \mathbf{D}$ is said to be dominating for E if almost every point of E is a nontangential limit of a sequence of points from Λ, and the set of all nontangential limits of Λ on $\mathbf{T}$ will be denoted by $NTL(\Lambda)$. The spaces $\mathbf{L}^1$ $(:=\mathbf{L}^1(\mathbf{T}))$, $\mathbf{H}^1$ $(:=\mathbf{H}^1(\mathbf{T}))$ and $\mathbf{H}^\infty$ $(:=\mathbf{H}^\infty(\mathbf{T}))$ are the usual Lebesgue and Hardy function spaces on $\mathbf{T}$, relative to the measure $\mathbf{m}$. It is easy to see that if $T \in (PB)(\mathcal{H})$, there exists a smallest number M such that (1) is valid for every polynomial p, and we denote the set of all $T \in (PB)(\mathcal{H})$ for which M is the smallest such number by $(PB)^M(\mathcal{H})$ (cf. [11]). If $T \in (PB)^M(\mathcal{H})$, it is easy to see that for any pair of vectors x and y $\in \mathcal{H}$ there exists a measure $\mu_{x,y}$ on $\mathbf{T}$ such that for every polynomial p,

$$\langle p(T)x\,,\,y\rangle = \int_{\mathbf{T}} p\,d\mu_{x,y}\ , \tag{2}$$

and the operator T is called *absolutely continuous* (notation $T \in (ACPB)^M(\mathcal{H})$) if for every pair x, y in $\mathcal{H}$ there exists an absolutely continuous measure $\mu_{x,y}$ satisfying (2) (with respect to $\mathbf{m}$).

For absolutely continuous polynomially bounded operators it is well known (cf. [11]) that there exists a unique unital, norm continuous algebra homomorphism

$$\Phi_T : \mathbf{H}^\infty \to \mathcal{A}_T$$

onto a weak* dense subalgebra of $\mathcal{A}_T$ such that Φ_T extends the Riesz-Dunford functional calculus, Φ_T is continuous if both $\mathbf{H}^\infty$ and $\mathcal{A}_T$ are given their weak*-topologies, and Φ_T is the adjoint of a bounded, linear, one to one map

$$\phi_T : \mathcal{Q}_T \to \mathbf{L}^1/\mathbf{H}^1_0.$$

Let us also recall (cf. [11]) that the class $\mathrm{A}^M(\mathcal{H})$ is the set of all $T \in (ACPB)^M(\mathcal{H})$ for which Φ_T is bounded below. In this case Φ_T is also a weak* homeomorphism between $\mathbf{H}^\infty$ onto $\mathcal{A}_T$, when $\mathbf{H}^\infty$ and $\mathcal{A}_T$ are given their weak*-topologies, and ϕ_T is onto.

For any f in $\mathbf{L}^1$, $[f]_{\mathbf{L}^1/\mathbf{H}^1_0}$ denotes the image of f in $\mathbf{L}^1/\mathbf{H}^1_0$ under the canonical projection from $\mathbf{L}^1$ onto $\mathbf{L}^1/\mathbf{H}^1_0$. If $\lambda \in \mathbf{D}$ and $\mathbf{P}_\lambda$ is the associated Poisson kernel on $\mathbf{T}$ (i.e., $\mathbf{P}_\lambda(t) := \frac{(1-|\lambda|^2)}{|1-\bar{\lambda}e^{it}|^2}$), we write

$$[C_\lambda]_T = \phi_T^{-1}([\mathbf{P}_\lambda]_{\mathbf{L}^1/\mathbf{H}^1_0}),$$

and it is easy to check that for any function h in $\mathbf{H}^\infty$,

$$\langle \Phi_T(h)\,,\,[C_\lambda]_T\rangle = h(\lambda).$$

If $T \in \mathbb{A}^M(\mathscr{H})$, then, as is customary, $\mathscr{E}_0(\mathscr{A}_T)$ denotes the set of all $[L]_T$ in $\mathscr{Q}_T$ for which there exist sequences $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$ in the unit ball of $\mathscr{H}$ such that

(i) $\lim_{n\to\infty} \|[L]_T - [x_n \otimes y_n]_T\| = 0$, and
(ii) $\lim_{n\to\infty}(\|[x_n \otimes w]_T\| + \|[w \otimes y_n]_T\|) = 0$ for any $w \in \mathscr{H}$,

and $\mathbb{A}^M(\mathscr{H})$ has property $\mathscr{X}_{0,\theta}$ ($\theta \in (0, 1]$) if $\mathscr{E}_0(\mathscr{A}_T)$ (which is (cf. [4]) absolutely convex and norm closed) contains the closed ball in $\mathscr{Q}_T$ centered at 0 with radius θ.

The following result comes from [11], and will be needed in the sequel.

Lemma 1. *Let* $T \in \mathbb{A}^M(\mathscr{H}) \cap C_{00}(\mathscr{H})$.

(i) *If* $\{x_n\}_{n=1}^\infty$ *is a sequence of vectors converging weakly to 0, then for any vector* $z \in \mathscr{H}$,

$$\lim_{n\to\infty}(\|[x_n \otimes z]_T\| + \|[z \otimes x_n]_T\|) = 0.$$

(ii) *If* $\lambda \in \sigma_e(T) \cap \mathbf{D}$, *then* $[C_\lambda]_T \in \mathscr{E}_0(\mathscr{A}_T)$.

Finally, we write, as is customary, $\mathbb{A}^M_{\aleph_0}(\mathscr{H})$ for the set of those operators T in $\mathbb{A}^M(\mathscr{H})$ such that for any doubly indexed sequence $\{[L_{ij}]_T\}_{i\geq 1;\ j\geq 1}$ of elements of $\mathscr{Q}_T$, there exist sequences $\{x_i\}_{i=1}^\infty$ and $\{y_j\}_{j=1}^\infty$ of vectors in $\mathscr{H}$ such that

$$[L_{ij}]_T = [x_i \otimes y_j]_T, \qquad 1 \leq i,\ 1 \leq j.$$

Now we may state the main result of this note.

Theorem 2. *Let* $T \in (PB)^M(\mathscr{H}) \cap C_{00}(\mathscr{H})$ *be such that* $\sigma(T) \cap \mathbf{D}$ *dominates* $\mathbf{T}$. *Then* $T \in \mathbb{A}^M_{\aleph_0}(\mathscr{H})$.

2. The details

In this section we prove Theorem 2.

Since for any function $h \in \mathbf{H}^\infty$, $h(\sigma(T)\cap\mathbf{D}) \subset \sigma(\Phi_T(h))$, it follows that Φ_T is bounded below, so $T \in \mathbb{A}^M(\mathscr{H})$. Thus by Theorem 3.7 of [2] it is sufficient to show that $\mathscr{A}_T$ has property $\mathscr{X}_{0,\theta}$ for some $\theta \in (0, 1]$. The following lemma is the main ingredient in showing this.

Lemma 3. *Suppose* ϵ, δ *are positive numbers,* f *is a nonnegative function in* $\mathbf{L}^1$, *and* $\{y_j\}_{j=1}^p$ *is a finite sequence of vectors in* $\mathscr{H}$. *Then there exists* $x \in \mathscr{H}$ *such that*

(i) $$\|\phi_T^{-1}([f]_{\mathbf{L}^1/\mathbf{H}^1_0}) - [x \otimes x]_T\| < \epsilon,$$

and

(ii) $\|x\| \leq 2\,\|f\|_1^{1/2}$, $\quad \|[x \otimes y_j]_T\| + \|[y_j \otimes x]_T\| < \delta$, $\quad j = 1, \ldots, p$.

Proof. Define $\Gamma_1 = (\sigma_p(T) \setminus \sigma_e(T))\cap\mathbf{D}$, $\Gamma_2 = (\sigma(T) \setminus (\sigma_p(T) \cup \sigma_e(T)))\cap\mathbf{D}$, $\widetilde{\Gamma}_1 = NTL(\Gamma_1)$, $\widetilde{\Gamma}_2 = NTL(\Gamma_2)$, and $\widetilde{\Gamma}_3 = NTL(\sigma_e(T)\cap\mathbf{D})$. First we consider $f\chi_{\widetilde{\Gamma}_1}$. By Lemma 1.2 of [3], there exist a finite sequence of positive numbers $\{\alpha_j^{(1)}\}_{j=1}^{n_1}$ and a finite sequence $\{\lambda_j^{(1)}\}_{j=1}^{n_1}$ of distinct points in Γ_1, such that

$$\sum_{j=1}^{n_1} \alpha_j^{(1)} \leq \|f\chi_{\widetilde{\Gamma}_1}\|_1$$

and

$$\|f\chi_{\widetilde{\Gamma}_1} - \sum_{j=1}^{n_1} \alpha_j^{(1)} P_{\lambda_j^{(1)}}\|_1 < \epsilon/5. \tag{3}$$

For each j choose a vector of norm one $x_j^{(1)} \in \ker(\lambda_j^{(1)} - T)$, and define $\mathscr{H}_1 = \operatorname{span}\{x_j^{(1)}\}_{j=1}^{n_1}$. Then $\mathscr{H}_1 \in \mathrm{Lat(T)}$, and by the choice of the sequence $\{\lambda_j^{(1)}\}_{j=1}^{n_1}$, the set $\{x_j^{(1)}\}_{j=1}^{n_1}$ is linearly independent. So $\dim \mathscr{H}_1 = n_1$, and $T_{\mathscr{H}_1}$ has the eigenvectors $\{x_j^{(1)}\}_{j=1}^{n_1}$ corresponding to the distinct eigenvalues $\{\lambda_j^{(1)}\}_{j=1}^{n_1}$. Therefore by Theorem 2.2 of [16] there exists $x^{(1)}$ in $\mathscr{H}_1$ with

$$\|x^{(1)}\| \le \|f\chi_{\widetilde{\Gamma}_1}\|_1^{1/2}$$

such that

$$[x^{(1)} \otimes x^{(1)}]_T = \sum_{j=1}^{n_1} \alpha_j^{(1)} [C_{\lambda_j^{(1)}}]_T.$$

Hence by (3),

$$\|\phi_T^{-1}([f\chi_{\widetilde{\Gamma}_2}]_{\mathbf{L}^1/\mathbf{H}_0^1}) - [x^{(1)} \otimes x^{(1)}]_T\| < \epsilon/5.$$

Since $\mathscr{H}_1$ is finite dimensional and invariant for T,

$$T_{\mathscr{H}\ominus\mathscr{H}_1} \in (ACPB)^M(\mathscr{H} \ominus \mathscr{H}_1),$$

$\sigma(T_{\mathscr{H}\ominus\mathscr{H}_1}) \cap \mathbf{D}$ dominates $\mathbf{T}$, $\Gamma_1 \setminus \{\lambda_j^{(1)}\}_{j=1}^{n_1} \subset (\sigma_p(T_{\mathscr{H}\ominus\mathscr{H}_1}) \setminus \sigma_e(T_{\mathscr{H}\ominus\mathscr{H}_1}))$, and $NTL(\Gamma_1 \setminus \{\lambda_j^{(1)}\}_{j=1}^{n_1}) = \widetilde{\Gamma}_1$. By the same argument as above (applied to $T_{\mathscr{H}\ominus\mathscr{H}_1}$), one can find a finite sequence of positive numbers $\{\alpha_j^{(2)}\}_{j=1}^{n_2}$, a finite sequence $\{\lambda_j^{(2)}\}_{j=1}^{n_2}$ of distinct points in $\Gamma_1 \setminus \{\lambda_j^{(1)}\}_{j=1}^{n_1}$, and a vector $x^{(2)}$ in $\mathscr{H} \ominus \mathscr{H}_1$ such that

$$\|f\chi_{\widetilde{\Gamma}_1} - \sum_{j=1}^{n_2} \alpha_j^{(2)} P_{\lambda_j^{(2)}}\|_1 < \epsilon/5, \tag{4}$$

$$[x^{(2)} \otimes x^{(2)}]_{T_{\mathscr{H}\ominus\mathscr{H}_1}} = \sum_{j=1}^{n_2} \alpha_j^{(2)} [C_{\lambda_j^{(2)}}]_{T_{\mathscr{H}\ominus\mathscr{H}_1}}, \tag{5}$$

and

$$\|x^{(2)}\| \le \|f\chi_{\widetilde{\Gamma}_1}\|_1^{1/2}.$$

Since $\mathscr{H}_1$ is invariant for T, by (5) it follows that

$$[x^{(2)} \otimes x^{(2)}]_T = \sum_{j=1}^{n_2} \alpha_j^{(2)} [C_{\lambda_j^{(2)}}]_T,$$

and taking into account (4), we obtain

$$\|\phi_T^{-1}([f\chi_{\widetilde{\Gamma}_1}]_{\mathbf{L}^1/\mathbf{H}_0^1}) - [x^{(2)} \otimes x^{(2)}]_T\| < \epsilon/5.$$

One can thus find by induction an orthogonal sequence $\{x^{(n)}\}_{n=1}^{\infty}$ such that for any positive integer n,

$$\|x^{(n)}\| \le \|f\chi_{\widetilde{\Gamma}_1}\|_1^{1/2}$$

and

$$\|\phi_T^{-1}([f\chi_{\widetilde{\Gamma}_1}]_{\mathbf{L}^1/\mathbf{H}_0^1}) - [x^{(n)} \otimes x^{(n)}]_T\| < \epsilon/5.$$

If M is large enough, $y^{(1)} := x^{(M)}$ satifies the inequalities

$$\|\phi_T^{-1}([f\chi_{\widetilde{\Gamma}_1}]_{\mathbf{L}^1/\mathbf{H}_0^1}) - [y^{(1)} \otimes y^{(1)}]_T\| < \epsilon/5\,, \tag{6}$$

$$\|y^{(1)}\| \le \|f\chi_{\widetilde{\Gamma}_1}\|_1^{1/2}\,,\ \|[y^{(1)} \otimes y_j]_T\| + \|[y_j \otimes y^{(1)}]_T\| < \delta/4\,, \quad j = 1, \dots, p. \tag{7}$$

By a similar argument as above (applied to $f\chi_{\widetilde{\Gamma}_2\setminus\widetilde{\Gamma}_1}$) one can obtain a vector $y^{(2)}$ such that

$$\|\phi_T^{-1}([f\chi_{\widetilde{\Gamma}_2\setminus\widetilde{\Gamma}_1}]_{\mathbf{L}^1/\mathbf{H}_0^1}) - [y^{(2)} \otimes y^{(2)}]_T\| < \epsilon/5\,, \tag{8}$$

$$\|y^{(2)}\| \le \|f\chi_{\widetilde{\Gamma}_2\setminus\widetilde{\Gamma}_1}\|_1^{1/2}\,, \tag{9}$$

$$\|[y^{(1)} \otimes y^{(2)}]_T\| + \|[y^{(2)} \otimes y^{(1)}]_T\| < \epsilon/5\,, \tag{10}$$

and

$$\|[y^{(2)} \otimes y_j]_T\| + \|[y_j \otimes y^{(2)}]_T\| < \delta/4\,, \quad j = 1, \dots, p. \tag{11}$$

Putting together (6)–(11), we get

$$\|y^{(1)} + y^{(2)}\| \le 2^{1/2}\,\|f\chi_{\widetilde{\Gamma}_1\cup\widetilde{\Gamma}_2}\|_1^{1/2}\,, \tag{12}$$

$$\|[(y^{(1)} + y^{(2)}) \otimes y_j]_T\| + \|[y_j \otimes (y^{(1)} + y^{(2)})]_T\| < \delta/2\,, \quad j = 1, \dots, p\,, \tag{13}$$

and

$$\|\phi_T^{-1}([f\chi_{\widetilde{\Gamma}_1\cup\widetilde{\Gamma}_2}]_{\mathbf{L}^1/\mathbf{H}_0^1}) - [(y^{(1)} + y^{(2)}) \otimes (y^{(1)} + y^{(2)})]_T\| < 4\epsilon/5. \tag{14}$$

Now we concentrate on $f\chi_{\mathbf{T}\setminus(\widetilde{\Gamma}_1\cup\widetilde{\Gamma}_2)}$. Since $\sigma_e(T)\cap\mathbf{D}$ dominates $\mathbf{T}\setminus(\widetilde{\Gamma}_1\cup\widetilde{\Gamma}_2)$, again by Lemma 1.2 of [3] one can find a finite sequence of positive numbers $\{\alpha_k\}_{k=1}^L$, and a finite sequence $\{\lambda_k\}_{k=1}^L \subset \sigma_e(T)$ such that

$$\sum_{k=1}^L \alpha_k \le \|f\chi_{\mathbf{T}\setminus(\widetilde{\Gamma}_1\cup\widetilde{\Gamma}_2)}\|_1\,,$$

and

$$\|\phi_T^{-1}([f\chi_{\mathbf{T}\setminus(\widetilde{\Gamma}_1\cup\widetilde{\Gamma}_2)}]_{\mathbf{L}^1/\mathbf{H}_0^1}) - \sum_{k=1}^L \alpha_k[C_{\lambda_k}]_T\| < \epsilon/20.$$

For each $k \in \{1, \dots, L\}$ let $\{x_n^{(k)}\}_{n=1}^\infty$ be a sequence of vectors in the unit ball of $\mathscr{H}$, converging weakly to 0 such that

$$\lim_{n\to\infty} \|[C_{\lambda_k}]_T - [x_n^{(k)} \otimes x_n^{(k)}]_T\| = 0.$$

By a standard argument (since $\lim\limits_{n\to\infty}(\|[x_n^{(k)} \otimes u]_T\| + \|[u \otimes x_n^{(k)}]_T\|) = 0$ for any u in $\mathscr{H}$, $k = 1, \dots, L$) one can choose inductively positive integers $\{n_k\}_{k=1}^L$ such that

$$y^{(3)} := \sum_{k=1}^L \alpha_k^{1/2} x_{n_k}^{(k)}$$

satisfies

$$(15) \qquad \|\phi_T^{-1}([f\chi_{\mathbf{T}\setminus(\widetilde{\Gamma}_1\cup\widetilde{\Gamma}_2)}]_{\mathbf{L}^1/\mathbf{H}^1_0}) - [y^{(3)} \otimes y^{(3)}]_T\| < \epsilon/10 ,$$

$$(16) \qquad \|y^{(3)}\| \leq 2^{1/2} \|f\chi_{\mathbf{T}\setminus(\widetilde{\Gamma}_1\cup\widetilde{\Gamma}_2)}\|_1^{1/2} ,$$

$$(17) \qquad \|[(y^{(1)} + y^{(2)}) \otimes y^{(3)}]_T\| + \|[y^{(3)} \otimes (y^{(1)} + y^{(2)})]_T\| < \epsilon/20,$$

and

$$(18) \qquad \|[y^{(3)} \otimes y_j]_T\| + \|[y_j \otimes y^{(3)}]_T\| < \delta/2 , \quad j = 1, \dots , L.$$

By (12)–(18) it follows easily that the vector $x := y^{(1)}+y^{(2)}+y^{(3)}$ satisfies (i) and (ii) above, and the lemma is proved.

The next step in the proof of Theorem 2 is to show that if f is a nonnegative function in $\mathbf{L}^1$ such that $\|f\|_1 \leq 1/2$, then $\phi_T^{-1}([f]_{\mathbf{L}^1/\mathbf{H}^1_0}) \in \mathscr{E}_0(\mathscr{A}_T)$. Once this has been shown, it will follow that if $f \in \mathbf{L}^1$ is such that $\|f\|_1 \leq 1/8$, then $\phi_T^{-1}([f]_{\mathbf{L}^1/\mathbf{H}^1_0}) \in \mathscr{E}_0(\mathscr{A}_T)$. Thus taking into account the facts that ϕ_T is invertible and $\|[f]_{\mathbf{L}^1/\mathbf{H}^1_0}\| \leq M \|\phi_T^{-1}([f]_{\mathbf{L}^1/\mathbf{H}^1_0})\|$ for any $f \in \mathbf{L}^1$, it will follow that $\mathscr{A}_T$ has property $\mathscr{X}_{0,1/8M}$, so we are done. To see that for any $f \in \mathbf{L}^1$ such that $\|f\|_1 \leq 1/8$, $\phi_T^{-1}([f]_{\mathbf{L}^1/\mathbf{H}^1_0}) \in \mathscr{E}_0(\mathscr{A}_T)$, pick two sequences of positive numbers $\{\epsilon_n\}_{n=1}^\infty$ and $\{\delta_n\}_{n=1}^\infty$ decreasing to 0, and a dense, countable subset $\{z_n\}_{n=1}^\infty$ in $\mathscr{H}$. By Lemma 2, one can find a sequence $\{x^{(n)}\}_{n=1}^\infty$ of vectors in the unit ball of $\mathscr{H}$ such that for every n,

$$\|\phi_T^{-1}([f]_{\mathbf{L}^1/\mathbf{H}^1_0}) - [x^{(n)} \otimes x^{(n)}]_T\| < \epsilon_n ,$$

and

$$\|[x^{(n)} \otimes z_k]_T\| + \|[z_k \otimes x^{(n)}]_T\| < \delta_n , \quad k = 1, \dots , n ,$$

so the sequence $\{x^{(n)}\}_{n=1}^\infty$ converges weakly to 0. Hence by Lemma 1, $\phi_T^{-1}([f]_{\mathbf{L}^1/\mathbf{H}^1_0}) \in \mathscr{E}_0(\mathscr{A}_T)$, and the proof of the theorem is complete.

Remarks. This paper constitutes part of the author's Ph.D. thesis written at Texas A&M University under the direction of Carl Pearcy. The referee has kindly pointed out that Jörg Eschmeier obtained a similar result in [10].

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