ISOMORPHISMS BETWEEN ENDOMORPHISM RINGS OF MODULES

KENNETH G. WOLFSON

(Communicated by Ken Goodearl)

ABSTRACT. We derive a necessary and sufficient condition that an isomorphism between the endomorphism rings of modules be induced by a semi-linear isomorphism of the modules. We require that at least one of the modules contain a non-zero free summand, generalizing work of Xu, who required both modules to be free.

INTRODUCTION

Xu in [6] derived a necessary and sufficient condition that an isomorphism between the endomorphism rings of free modules be induced by a semi-linear isomorphism of the modules.

In this paper we generalize this result to any pair of modules, at least one of which contains a non-zero free summand (Theorem 2.1). Our proof utilizes an earlier result which we state and prove as Lemma 1.2.

1. DEFINITION AND PRELIMINARIES

All rings have identity elements. The left unital module A over the ring F will be denoted (F, A) and its endomorphism ring (operating on the right of A) will be denoted $E(F, A)$, or simply E . The right E module A is denoted (A, E) . The statement that (F, A, E) is a bimodule indicates that A is to be considered as a left F module and a right E module. A summand of A which is free on one generator will be called a “point” of A . If A has a point Fa , then $A = Fa \oplus K$ and we have an associated idempotent e in $E(F, A)$ defined by $ae = a$, $Ke = 0$.

The arguments of Baer [1, Proposition 5, p. 176] with minor modifications can be used to prove:

Lemma 1.1. *If (F, A) contains a point, and e is an associated idempotent, the bimodule (F, A, E) is isomorphic to the bimodule (eEe, eE, E) in the sense that:*

- (1) *there is a semi-linear isomorphism (ρ, γ) of the left modules (F, A) and (eEe, eE) .*
- (2) *γ is an E -isomorphism of the right modules (A, E) and (eE, E) .*

Received by the editors July 16, 1993 and, in revised form, October 25, 1993.

1991 *Mathematics Subject Classification.* Primary 16S50; Secondary 15A04, 16D99.

©1995 American Mathematical Society

The following result is the key to relating module and endomorphism ring isomorphisms, when at least one of the modules contains a point. Its proof is implicit in section 4 of [2]. See also [3, Theorem 2.10]. We give the proof outlined in [4, Lemma 1.1]. See also [5, Theorem 2.1].

Lemma 1.2. *Let (F, A) and (G, B) be modules with (F, A) containing a point, and let ϕ be a ring isomorphism of $E(F, A)$ upon $E(G, B)$. Then ϕ is induced by a semi-linear module isomorphism of (F, A) upon (G, B) if, and only if, there is a point of A with associated idempotent $e \in E(F, A)$ such that Be^ϕ is a point of B .*

Proof. Let $e^\phi = f$ and assume Ae and Bf are points of A and B respectively, and consider the sequence of bimodule maps:

$$(F, A, E_1) \xrightarrow{\rho_1} (eE_1e, eE_1, E_1) \xrightarrow{\phi} (fE_2f, fE_2, E_2) \xrightarrow{\rho_2} (G, B, E_2)$$

where $E_1 = E(F, A)$, $E_2 = E(G, B)$, ρ_1 is the bimodule isomorphism of Lemma 1.1, ρ_2 is the (inverse of) the bimodule isomorphism of Lemma 1.1 applied to $[G, B, E(G, B)]$, and ϕ is the bimodule map induced by the ring isomorphism ϕ of E_1 onto E_2 which clearly maps eE_1 onto fE_2 and eE_1e onto fE_2f . Then $s = \rho_1\phi\rho_2$ is easily seen to be a semi-linear module isomorphism of (F, A) onto (G, B) . Now let $b \in B$, $\sigma \in E(F, A)$, $a^s = b$, ($a \in A$). Then, $(a\sigma)^s = (a\sigma)^{\rho_1\phi\rho_2} = (a^{\rho_1}\sigma)^{\phi\rho_2} = (a^{\rho_1}\sigma^\phi)^{\rho_2} = a^s\sigma^\phi$, or $(b^{s^{-1}}\sigma)^s = b\sigma^\phi$ for all $b \in B$. Thus $\sigma^\phi = s^{-1}\sigma s$.

Conversely if ϕ is induced by a semi-linear module isomorphism, $A\sigma$ and $B\sigma^\phi$ are semi-linearly isomorphic for each $\sigma \in E(F, A)$. Hence if Ae is any point of A , then Be^ϕ must be a point of B .

2. THE MAIN RESULT

Let (F, A) be a free module with basis $\{a_i\}$, $i \in \Gamma$. The set of endomorphisms $\{e_i\} \subseteq E(F, A)$ defined by $a_ie_i = a_i$, $a_je_i = 0$, $i \neq j \in \Gamma$, is called the set of associated idempotents [to the basis $\{a_i\}$].

Xu's results (Theorems 1 and 2 of [6]) may now be stated as follows:

Let (F, A) and (G, B) be free modules, and let ϕ be a ring isomorphism of $E(F, A)$ upon $E(G, B)$. Then ϕ is induced by a semi-linear module isomorphism of (F, A) upon (G, B) if, and only if, there exist bases $\{a_i\}$, $i \in \Gamma$, $\{b_i\}$, $i \in \Gamma'$, of A and B respectively, with associated idempotents $\{e_i\}$ and $\{f_i\}$ such that

- (1) $fe^\phi E(G, B) = fE(G, B)$ and
- (2) $E(G, B)fe^\phi = E(G, B)e^\phi$

for some $e \in \{e_i\}$ and $f \in \{f_i\}$.

We generalize Xu's result as follows:

Theorem 2.1. *Let (F, A) and (G, B) be modules with (F, A) containing a point, and let ϕ be a ring isomorphism of $E(F, A)$ upon $E(G, B)$. Then ϕ is induced by a semi-linear module isomorphism of (F, A) upon (G, B) if, and only if, there exists a point of A and a point of B with associated idempotents $e \in E(F, A)$, $f \in E(G, B)$ respectively such that:*

- (1) $fe^\phi E(G, B) = fE(G, B)$ and
- (2) $E(G, B)fe^\phi = E(G, B)e^\phi$.

Proof. To prove sufficiency, we note that (2) implies $BE(G, B)fe^\phi = BE(G, B)e^\phi$. Since $BE = B$, we have $Bfe^\phi = Be^\phi$. But $Bf = Gc$, $c \in B$, a free generator, hence $Gce^\phi = Be^\phi$, and letting $ce^\phi = b$ gives $Gb = Be^\phi$. By Lemma 1.2 we need only show that Gb is free on the generator b . From (1) $cfe^\phi E(G, B) = cfE(G, B)$. But $cf = c$ yields $ce^\phi E(G, B) = cE(G, B)$. Now let $\alpha \in G$ with $\alpha b = \alpha ce^\phi = 0$. This implies $\alpha cE(G, B) = 0$, or $\alpha c = 0$ [$1 \in E(G, B)$]. Since c is a free generator, $\alpha = 0$.

Conversely, for any point of A , let e be an associated idempotent. By Lemma 1.2, Be^ϕ is a point of B . Put $f = e^\phi$, and (1) and (2) are obviously satisfied.

REFERENCES

1. R. Baer, *Linear algebra and projective geometry*, Academic Press, New York, 1952.
2. W. N. Franzen and P. Schultz, *The endomorphism ring of a locally free module*, J. Austral. Math. Soc. Ser. A **35** (1983), 308–326.
3. A. V. Mikhalev, *Isomorphisms of ring endomorphisms of modules that are close to free*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **44** (1989), 20–27.
4. K. G. Wolfson, *Isomorphisms of prime Goldie semi-principal left ideal rings. II*, Canad. Math. Bull. **31** (1988), 374–379.
5. ———, *Anti-isomorphisms of endomorphism rings of locally free modules*, Math. Z. **202** (1989), 151–159.
6. Y. Xu, *Isomorphisms between endomorphism rings of free modules*, Sci. China Ser. A **35** (1992), 769–779.

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903
 Current address: 4565 Cocoplum Way, Delray Beach, Florida 33445