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THE EMBEDDING THEOREM FOR THE BESOV AND TRIEBEL-LIZORKIN SPACES ON SPACES OF HOMOGENEOUS TYPE

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ABSTRACT. In this note the classical embedding theorem for the Besov and Triebel-Lizorkin spaces on R^n is generalized to the Besov and Triebel-Lizorkin spaces on spaces of homogeneous type. The proof is new even for R^n case.

INTRODUCTION

Suppose a function ϕ satisfies the conditions: (i) $\phi \in S$; (ii) $\operatorname{Supp}\hat{\phi} \subseteq \{\xi \in R^n: \frac{1}{2} \le |\xi| \le 2\}$; (iii) $|\hat{\phi}(\xi)| \ge c > 0$ if $\frac{3}{5} \le |\xi| \le \frac{5}{3}$. The Besov and Triebel-Lizorkin spaces can be defined as follows:

$$\|f\|_{\dot{B}_p^{\alpha,q}} = \left\{ f \in S'/P: \left\{ \sum_{k\in Z} (2^{k\alpha}\|\phi_k * f\|_p)^q \right\}^{1/q} < \infty \right\}$$

for $0 < p, q \le \infty$, $\alpha \in R$, and

$$\|f\|_{\dot{F}_p^{\alpha,q}} = \left\{ f \in S'/P: \left\| \left\{ \sum_{k\in Z} (2^{k\alpha}|\phi_k * f|)^q \right\}^{1/q} \right\|_p < \infty \right\}$$

for $0 < p < \infty$, $0 < q \le \infty$, and $\alpha \in R$, where P is the collection of all polynomials and $\phi_k(x) = 2^{kn}\phi(2^k x)$.

The classical embedding theorem for these spaces is given by the following.

Theorem A. *Suppose* $-\infty < s_1 < s_0 < \infty$, $0 < p_0, p_1 < \infty$, $0 < q_0, q_1, q \le \infty$, *and* $s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}$. *Then*

(i) $$\dot{B}_{p_0}^{s_0,q} \to \dot{B}_{p_1}^{s_1,q},$$

(ii) $$\dot{F}_{p_0}^{s_0,q_0} \to \dot{F}_{p_1}^{s_1,q_1}.$$

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We remark that since $\dot{F}_p^{0,2} = L^p$ for $1 < p < \infty$ and $\dot{F}_p^{\alpha,2} = I_\alpha(L^p)$, where I_α, $0 < \alpha < n$, is the Riesz potential, the theorem above includes the classical Sobolev embedding theorem. See [P] and [T] for more details.

In this note we will generalize the classical embedding theorem for the Besov and Triebel-Lizorkin spaces on R^n to the Besov and Triebel-Lizorkin spaces on spaces of homogeneous type in the sense of Coifman and Weiss ([CW]).

We begin by recalling the definitions necessary for introducing the Besov and Triebel-Lizorkin spaces on spaces of homogeneous type. A quasi-metric d on a set X is a function $d\colon X \times X \to [0, \infty]$ satisfying:

$$\text{(1.1)}\quad \begin{aligned} &\text{(i)} && d(x, y) = 0 \quad \text{if and only if } x = y, \\ &\text{(ii)} && d(x, y) = d(y, x) \quad \text{for all } x, y \in X, \\ &\text{(iii)} && \text{there exists a constant } A < \infty \text{ such that for all } x, y, z \in X, \\ &&& d(x, y) \le A[d(x, z) + d(z, y)]. \end{aligned}$$

Any quasi-metric defines a topology, for which the balls $B(x, r) = \{y \in X\colon d(y, x) < r\}$ form a base. However, the balls themselves need not be open when $A > 1$.

Definition 1.2 ([CW]). A space of homogeneous type (X, d, μ) is a set X together with a quasi-metric d and a nonnegative measure μ on X such that $\mu(B(x, r)) < \infty$ for all $x \in X$ and all $r > 0$, and there exists $A' < \infty$ such that for all $x \in X$ and all $r > 0$,

$$\mu(B(x, 2r)) \le A'\mu(B(x, r)). \tag{1.3}$$

Here μ is assumed to be defined on a σ-algebra which contains all Borel sets and all balls $B(x, r)$.

Macias and Segovia [MS] have shown that one can replace d by another quasi-metric ρ such that there exist $c < \infty$ and some θ, $0 < \theta < 1$,

$$\rho(x, y) \approx \inf\{\mu(B)\colon B \text{ is a ball containing } x \text{ and } y\}, \tag{1.4}$$

$$|\rho(x, y) - \rho(x', y)| \le c\rho(x, x')^\theta[\rho(x, y) + \rho(x', y)]^{1-\theta} \quad \text{for all } x, x', \text{ and } y \in X. \tag{1.5}$$

There are many interesting examples of spaces of homogeneous type. For instance, any C^∞ compact Riemannian manifold with the Riemannian metric and volume and the boundary of any bounded Lipschitz domain in R^n with the induced Euclidean metric and the Lebesgue measure are spaces of homogeneous type. See [Ch] for more examples. The regularity exponent θ depends on spaces of homogeneous type, for example, $\theta = 1$ for any bounded Lipschitz domain in R^n. We will suppose that $\mu(X) = \infty$ and $\mu(\{x\}) = 0$ for all $x \in X$. These hypotheses allow us to construct an approximation to the identity (see [HS]).

Definition 1.6. A sequence $(S_k)_{k\in Z}$ of operators is called to be an approximation to the identity if $S_k(x, y)$, the kernels of S_k, are functions from $X \times X$ into $\mathscr{C}$ such that there exists a constant C, some $0 < \varepsilon \le \theta$, and some $c < \infty$, for all $k \in Z$ and all x, x', y, and $y' \in X$,

(i) $$S_k(x, y) = 0 \quad \text{if } \rho(x, y) \ge c2^{-k} \text{ and } \|S_k\|_\infty \le C2^k,$$

(ii) $$|S_k(x, y) - S_k(x', y)| \le C2^{k(1+\varepsilon)}\rho(x, x')^\varepsilon,$$

(iii) $$|S_k(x, y) - S_k(x, y')| \le C2^{k(1+\varepsilon)}\rho(y, y')^\varepsilon,$$

(iv) $$|[S_k(x, y) - S_k(x, y')] - [S_k(x', y) - S_k(x', y')]| \le C\rho(x, x')^\varepsilon \rho(y, y')^\varepsilon 2^{k(1+2\varepsilon)},$$

(v) $$\int_X S_k(x, y)\, d\mu(y) = 1,$$

(vi) $$\int_X S_k(x, y)\, d\mu(x) = 1.$$

See [DJS] for the existence of such a sequence of operators; there all conditions are introduced and checked except condition (iv) in 1.6. It is easy to see that the same construction in [DJS] satisfies condition (iv). To define the Besov and Triebel-Lizorkin spaces on spaces of homogeneous type we need the following definition (see [HS]).

Definition 1.7. Fix two exponents $0 < \beta \le \theta$ and $\gamma > 0$. A function f defined on X is said to be a strong smooth molecule of type (β, γ) centered at $x_0 \in X$ with width $d > 0$, if f satisfies the following conditions:

(i) $$|f(x)| \le c\frac{d^\gamma}{(d + \rho(x, x_0))^{1+\gamma}},$$

(ii) $$|f(x) - f(x')| \le c\left[\frac{\rho(x, x')}{d + \rho(x, x_0)}\right]^\beta \frac{d^\gamma}{(d + \rho(x, x_0))^{1+\gamma}}$$

for $\rho(x, x') \le \frac{1}{2A}(d + \rho(x, x_0))$,

(iii) $$\int_X f(x)\, d\mu(x) = 0.$$

This definition was first introduced in [M] for the case $X = R^n$ with condition (ii) in (1.7) replaced by

(1.8) $$|f(x) - f(x')| \le c\left[\frac{\rho(x, x')}{d}\right]^\beta \left[\frac{d^\gamma}{(d + \rho(x, x_0))^{1+\gamma}} + \frac{d^\gamma}{(d + \rho(x', x_0))^{1+\gamma}}\right].$$

The collection of all strong smooth molecules of type (β, γ) centered at $x_0 \in X$ with width $d > 0$ will be denoted by $\mathscr{M}^{(\beta,\gamma)}(x_0, d)$. If $f \in \mathscr{M}^{(\beta,\gamma)}(x_0, d)$, the norm of f in $\mathscr{M}^{(\beta,\gamma)}(x_0, d)$ is then defined by

(1.9) $$f\|_{\mathscr{M}^{(\beta,\gamma)}(x_0, d)} = \inf\{c \ge 0\colon \text{ (i) and (ii) in (1.7) hold}\}.$$

Now we fix a point $x_0 \in X$ and denote the class of all $f \in \mathscr{M}^{(\beta,\gamma)}(x_0, 1)$ by $\mathscr{M}^{(\beta,\gamma)}$. It is easy to see that $\mathscr{M}^{(\beta,\gamma)}$ is a Banach space under the norm $\|f\|_{\mathscr{M}^{(\beta,\gamma)}} < \infty$. Just as the space of distributions $\mathscr{S}'$ is defined on R^n, the dual space $(\mathscr{M}^{(\beta,\gamma)})'$ consists of all linear functionals $\mathscr{L}$ from $\mathscr{M}^{(\beta,\gamma)}$ to $\mathscr{C}$ with the property that there exists a finite constant c such that for all $f \in \mathscr{M}^{(\beta,\gamma)}$, $|\mathscr{L}(f)| \le c\|f\|_{\mathscr{M}^{(\beta,\gamma)}}$. We denote the natural pairing of elements $h \in (\mathscr{M}^{(\beta,\gamma)})'$ and $f \in \mathscr{M}^{(\beta,\gamma)}$ by $\langle h, f\rangle$. It is also easy to see that for $x_1 \in X$ and $d > 0$, $\mathscr{M}^{(\beta,\gamma)}(x_1, d) = \mathscr{M}^{(\beta,\gamma)}$ with equivalent norms. Thus, $\langle h, f\rangle$ is well defined for all $h \in (\mathscr{M}^{(\beta,\gamma)})'$ and all $f \in \mathscr{M}^{(\beta,\gamma)}(x_1, d)$ with $x_1 \in X$ and

$d > 0$. In [HS] the Besov and Triebel-Lizorkin spaces on spaces of homogeneous type were introduced by use of the sequence of operators $(D_k)_{k\in Z}$ where $D_k(f)(x) = \int_X D_k(x, y)f(y)\,d\mu(y)$ and $D_k(x, y) = S_k(x, y) - S_{k-1}(x, y)$ and $(S_k)_{k\in Z}$ is the approximation to the identity defined in (1.6). More precisely, the Besov space $\dot{B}_p^{\alpha,q}$ for $-\varepsilon < \alpha < \varepsilon$ and $1 \le p, q \le \infty$ is the collection of all $f \in (\mathcal{M}^{(\beta,\gamma)})'$ with $0 < \beta, \gamma < \varepsilon$ such that

$$\|f\|_{\dot{B}_p^{\alpha,q}} = \left\{\sum_{k\in Z}(2^{k\alpha}\|D_k(f)\|_p)^q\right\}^{1/q} < \infty. \tag{1.10}$$

The Triebel-Lizorkin space $\dot{F}_p^{\alpha,q}$ for $-\varepsilon < \alpha < \varepsilon$ and $1 < p, q < \infty$ is the collection of all $f \in (\mathcal{M}^{(\beta,\gamma)})'$ with $0 < \beta, \gamma < \varepsilon$ such that

$$\|f\|_{\dot{F}_p^{\alpha,q}} = \left\|\left\{\sum_{k\in Z}(2^{k\alpha}|D_k(f)|)^q\right\}^{1/q}\right\|_p < \infty. \tag{1.11}$$

In this note we prove the following embedding theorem for the Besov and Triebel-Lizorkin spaces on spaces of homogeneous type.

Theorem. *Suppose* $-\varepsilon < s_1 < s_0 < \varepsilon$. *Then*

(i) $\dot{B}_{p_0}^{s_0,q} \to \dot{B}_{p_1}^{s_1,q}$ *for* $1 \le q \le \infty$, $1 \le p_0, p_1 \le \infty$, *and* $-\varepsilon < s_0 - \frac{1}{p_0} = s_1 - \frac{1}{p_1} < \varepsilon$;

(ii) $\dot{F}_{p_0}^{s_0,q_0} \to \dot{F}_{p_1}^{s_1,q_1}$ *for* $1 < p_0, p_1, q_0, q_1 < \infty$, *and* $-\varepsilon < s_0 - \frac{1}{p_0} = s_1 - \frac{1}{p_1} < \varepsilon$.

2. Proof of the Theorem

The proof of the classical embedding Theorem A depends on the Fourier transform. To be precise, if ϕ is a function as in the definition of the Besov and Triebel-Lizorkin spaces and $f \in \mathcal{S}'/\mathcal{P}$, then, using the Fourier transform, we have the inequality

$$\|\phi_k * f\|_\infty \le c2^{kn/p}\|\phi_k * f\|_p, \tag{2.1}$$

which, by Hölder's inequality, yields

$$\|\phi_k * f\|_{p_1} \le c2^{kn(1/p_0 - 1/p_1)}\|\phi_k * f\|_{p_0}. \tag{2.2}$$

This gives (i) of Theorem A. Similarly, the proof of (ii) of Theorem A also needs (2.1). Since there is no Fourier transform on spaces of homogeneous type, we need a new idea to prove the Theorem. Our starting point is to use the Calderon type reproducing formula which was obtained in [HS]. Since we never use the Fourier transform, our method is new even for the case of R^n.

The Calderon type reproducing formula ([HS]). Suppose $\{D_k\}$ is the family of operators used in the definitions of the Besov and Triebel-Lizorkin spaces. Then there exists a sequence of operators $\{\widetilde{D}_k\}$ such that for all $f \in (\mathcal{M}^{(\beta,\gamma)})'$

$$f = \sum_k \widetilde{D}_k D_k(f) \tag{2.3}$$

where the series converges in $(\mathscr{M}^{(\beta',\gamma')})'$ with $\beta' > \beta$ and $\gamma' > \gamma$. Moreover, $\widetilde{D}_k(x, y)$, the kernel of $\widetilde{D}_k$, satisfies the following estimates: For $0 < \varepsilon' < \varepsilon$ there exists a constant c such that

$$\text{(i)} \qquad |\widetilde{D}_k(x, y)| \le c\frac{2^{-k\varepsilon'}}{(2^{-k} + \rho(x, y))^{1+\varepsilon'}},$$

$$\text{(ii)} \qquad |\widetilde{D}_k(x, y) - \widetilde{D}_k(x', y)| \le c\left[\frac{\rho(x, x')}{(2^{-k} + \rho(x, y))}\right]^{\varepsilon'} \frac{2^{-k\varepsilon'}}{(2^{-k} + \rho(x, y))^{1+\varepsilon'}}$$

for $\rho(x, x') \le \frac{1}{2A}(2^{-k} + \rho(x, y))$,

$$\text{(iii)} \qquad \int \widetilde{D}_k(x, y)d\mu(x) = 0 \quad \text{for all } y \in X.$$

Lemma 2.4. *For $0 < \varepsilon' < \varepsilon$ there exists a constant c such that $D_k\widetilde{D}_j(x, y)$, the kernel of $D_k\widetilde{D}_j$, satisfies the following estimate:*

$$(2.5) \qquad |D_k\widetilde{D}_j(x, y)| \le c2^{-|k-j|\varepsilon'} \frac{(2^{-k} \vee 2^{-j})^{\varepsilon'}}{\{(2^{-k} \vee 2^{-j}) + \rho(x, y)\}^{1+\varepsilon'}}$$

where $a \vee b$ denotes the maximum of a and b.

We now prove (2.5). Notice that the kernel of D_k satisfies the conditions (i)–(iv) in (1.6) and $\int D_k(x, y)d\mu(y) = 0$, and $\int D_k(x, y)d\mu(x) = 0$. Consider first that $k \ge j$ and $2cA2^{-k} \le 2^{-j}$ or $k \ge j$ and $\rho(x, y) \ge 2cA2^{-j}$. Then

$$|D_k\widetilde{D}_j(x, y)| = \left|\int D_k(x, z)\widetilde{D}_j(z, y)d\mu(z)\right|$$
$$= \left|\int D_k(x, z)[\widetilde{D}_j(z, y) - \widetilde{D}_j(x, y)]d\mu(z)\right|$$

since $\int D_k(x, z)d\mu(z) = 0$

$$\le c\int |D_k(x, z)| \frac{\rho(x, z)^{\varepsilon'}}{(2^{-j} + \rho(x, y))^{\varepsilon'}} \frac{2^{-j\varepsilon'}}{\{2^{-j} + \rho(x, y)\}^{1+\varepsilon'}} d\mu(z)$$

by the fact that $\rho(x, z) \le c2^{-k} \le \frac{1}{2A}(2^{-j} + \rho(x, y))$ and the smoothness of the kernel of $\widetilde{D}_j$

$$\le c\frac{2^{-k\varepsilon'}}{\{2^{-j} + \rho(x, y)\}^{1+\varepsilon'}}$$

by the size condition of the kernel of D_k in (i) of (1.6)

$$\le c2^{-(k-j)\varepsilon'}\frac{2^{-j\varepsilon'}}{\{2^{-j} + \rho(x, y)\}^{1+\varepsilon'}}$$

which shows (2.5) for the case where $k \ge j$ and $2cA2^{-k} \le 2^{-j}$ or $k \ge j$, and $\rho(x, y) \ge 2cA2^{-j}$.

Consider now the case that $k \ge j$, $2cA2^{-k} > 2^{-j}$ and $\rho(x, y) < 2cA2^{-j}$. Then

$$|D_k\widetilde{D}_j(x, y)| = \left|\int D_k(x, z)\widetilde{D}_j(z, y)d\mu(z)\right| \le c2^j$$

by the size conditions on the kernel of D_k and $\widetilde{D}_j$

$$\leq c2^{(j-k)\varepsilon'}\frac{2^{-j\varepsilon'}}{\{2^{-j}+\rho(x,y)\}^{1+\varepsilon'}}$$

by the facts that $2^{(k-j)\varepsilon'} \leq (2cA)^{\varepsilon'}$ and $\rho(x,y) < 2cA2^{-j}$, which together with the above estimate shows (2.5) for the case $k \geq j$. The proof of (2.5) for the case $k < j$ is similar and easier.

We may assume that ε', the regularity exponent in the Calderon type reproducing formula, satisfies $-\varepsilon' < s_1 < s_0 < \varepsilon'$ and $-\varepsilon' < s_0 - \frac{1}{p_0} = s_1 - \frac{1}{p_1} < \varepsilon'$. Now we prove (i) of the Theorem. Suppose that $f \in \dot{B}_{p_0}^{s_0,q}$. By the Calderon type reproducing formula we have

$$\begin{aligned}\|D_k(f)\|_{p_1} &= \left\|\sum_j D_k\widetilde{D}_jD_j(f)\right\|_{p_1} \leq \sum_j \|D_k\widetilde{D}_jD_j(f)\|_{p_1} \\ &\leq c\sum_j 2^{-|k-j|\varepsilon'}(2^{-k}\vee 2^{-j})^{(1/p_1-1/p_0)}\|D_j(f)\|_{p_0}\end{aligned}$$

by Young's inequality and (2.5). Thus,

$$\begin{aligned}\|f\|_{\dot{B}_{p_1}^{s_1,q}} &= \left\{\sum_{k\in Z}(2^{ks_1}\|D_k(f)\|_{p_1})^q\right\}^{1/q} \\ &\leq c\left\{\sum_{k\in Z}(2^{ks_1}\sum_j 2^{-|k-j|\varepsilon'}(2^{-k}\vee 2^{-j})^{(1/p_1-1/p_0)}\|D_j(f)\|_{p_0})^q\right\}^{1/q} \\ &\leq c\left\{\sum_{k\in Z}\left(\sum_{j>k}2^{-|k-j|\varepsilon'}2^{ks_1}2^{-k(1/p_1-1/p_0)}\|D_j(f)\|_{p_0}\right)^q\right\}^{1/q} \\ &\quad + c\left\{\sum_{k\in Z}\left(\sum_{j\leq k}2^{-|k-j|\varepsilon'}2^{ks_1}2^{-j(1/p_1-1/p_0)}\|D_j(f)\|_{p_0}\right)^q\right\}^{1/q} \\ &\leq c\left\{\sum_{k\in Z}\left(\sum_{j>k}2^{-|k-j|\varepsilon'}2^{ks_0}\|D_j(f)\|_{p_0}\right)^q\right\}^{1/q} \\ &\quad + c\left\{\sum_{k\in Z}\left(\sum_{j\leq k}2^{-|k-j|\varepsilon'}2^{ks_1}2^{-j(s_1-s_0)}\|D_j(f)\|_{p_0}\right)^q\right\}^{1/q}\end{aligned}$$

by the fact that $\frac{1}{p_1} - \frac{1}{p_0} = s_1 - s_0$

$$\leq c\left\{\sum_{j\in Z}(2^{js_0}\|D_j(f)\|_{p_0})^q\right\}^{1/q}$$

by Hölder's inequality and the fact that

$$\sum_{k\in Z} 2^{-|k-j|\varepsilon'}2^{(k-j)s_0} + \sum_{k\in Z} 2^{-|k-j|\varepsilon'}2^{(k-j)s_1} + \sum_{j\in Z} 2^{-|k-j|\varepsilon'}2^{(k-j)s_0} + \sum_{j\in Z} 2^{-|k-j|\varepsilon'}2^{(k-j)s_1} < \infty$$

if $-\varepsilon' < s_1 < s_0 < \varepsilon'$

$$= c\|f\|_{\dot{B}^{s_0,q}_{p_0}}.$$

To prove (ii) of the Theorem, by the homogeneity, it suffices to take $\|f\|_{\dot{F}^{s_0,q_0}_{p_0}} = 1$. By the Calderon type reproducing formula, Hölder's inequality, and the estimate in (2.5),

(2.6)

$$\begin{aligned}
|D_k(f)| &= \left|\sum_j D_k\widetilde{D}_jD_j(f)\right| \le \sum_j |D_k\widetilde{D}_jD_j(f)| \\
&\le c\sum_j 2^{-|k-j|\varepsilon'}(2^{-k}\vee 2^{-j})^{-1/p_0}\|D_j(f)\|_{p_0} \\
&\le c\sum_j 2^{-|k-j|\varepsilon'}(2^{-k}\vee 2^{-j})^{-1/p_0}2^{-js_0}\left\|\left\{\sum_{k\in Z}(2^{ks_0}|D_k(f)|)^{q_0}\right\}^{1/q_0}\right\|_{p_0} \\
&\le c\sum_j 2^{-|k-j|\varepsilon'}(2^{k}\vee 2^{j})^{1/p_0}2^{-js_0}.
\end{aligned}$$

Therefore, for any fixed integer N

$$\begin{aligned}
&\left\{\sum_{-\infty}^{N}(2^{ks_1}|D_k(f)|)^{q_1}\right\}^{1/q_1} \\
&\quad\le c\left\{\sum_{-\infty}^{N}\left(2^{ks_1}\sum_j 2^{-|k-j|\varepsilon'}(2^{k}\vee 2^{j})^{1/p_0}2^{-js_0}\right)^{q_1}\right\}^{1/q_1} \\
&\quad\le c\left\{\sum_{-\infty}^{N}\left(2^{ks_1}\sum_{j>k} 2^{-|k-j|\varepsilon'}2^{j/p_0}2^{-js_0}\right)^{q_1}\right\}^{1/q_1} \qquad (2.7)\\
&\qquad+\left\{\sum_{-\infty}^{N}\left(2^{ks_1}\sum_{j\le k} 2^{-|k-j|\varepsilon'}2^{k/p_0}2^{-js_0}\right)^{q_1}\right\}^{1/q_1} \\
&\quad\le c\left\{\sum_{-\infty}^{N}(2^{k/p_1})^{q_1}\right\}^{1/q_1}
\end{aligned}$$

since $s_0-\frac{1}{p_0} = s_1-\frac{1}{p_1}$ and $\sum_{j>k} 2^{-|k-j|\varepsilon'}2^{(j-k)(1/p_0-s_0)}+\sum_{j\le k} 2^{-|k-j|\varepsilon'}2^{(k-j)s_0} <$

∞ if $-\varepsilon' + \frac{1}{p_0} < s_0 < \varepsilon' \le c2^{N/p_1}$. On the other hand,

$$\left\{\sum_N^\infty (2^{ks_1}|D_k(f)|)^{q_1}\right\}^{1/q_1} = \left\{\sum_N^\infty (2^{k(s_1-s_0)}2^{ks_0}|D_k(f)|)^{q_1}\right\}^{1/q_1}$$

$$\le \left\{\sum_N^\infty (2^{k(s_1-s_0)})^{q_1}\right\}^{1/q_1} \left\{\sum_{k\in Z} (2^{ks_0}|D_k(f)|)^{q_0}\right\}^{1/q_0} \tag{2.8}$$

$$\le c2^{N(s_1-s_0)} \left\{\sum_{k\in Z} (2^{ks_0}|D_k(f)|)^{q_0}\right\}^{1/q_0}$$

since $s_1 < s_0$

$$\le c2^{N(1/p_1-1/p_0)} \left\{\sum_{k\in Z} (2^{ks_0}|D_k(f)|)^{q_0}\right\}^{1/q_0}$$

since $s_0 - 1/p_0 = s_1 - 1/p_1$. We now obtain

$$\begin{aligned}
\|f\|^{p_1}_{\dot F^{s_1,q_1}_{p_1}} &= p_1 \int_0^\infty t^{p_1-1} \left|\left\{ \left(\sum_{k\in Z} (2^{ks_1}|D_k(f)|)^{q_1}\right)^{1/q_1} > t\right\}\right| dt \\
&\le p_1 \sum_{-\infty}^\infty \int_{2c2^{N/p_1}}^{2c^{(N+1)/p_1}} t^{p_1-1} \left|\left\{ \left(\sum_{k\in Z} (2^{ks_1}|D_k(f)|)^{q_1}\right)^{1/q_1} > t\right\}\right| dt \\
&\le p_1 \sum_{-\infty}^\infty \int_{2c2^{N/p_1}}^{2c2^{(N+1)/p_1}} t^{p_1-1} \left|\left\{ \left(\sum_{-\infty}^N (2^{ks_1}|D_k(f)|)^{q_1}\right)^{1/q_1} \right.\right. \\
&\qquad\qquad \left.\left. + \left(\sum_N^\infty (2^{ks_1}|D_k(f)|)^{q_1}\right)^{1/q_1} > t\right\}\right| dt \\
&\le p_1 \sum_{-\infty}^\infty \int_{2c2^{N/p_1}}^{2c2^{(N+1)/p_1}} t^{p_1-1} \left|\left\{ \left(\sum_N^\infty (2^{ks_1}|D_k(f)|)^{q_1}\right)^{1/q_1} > \frac{1}{2}t\right\}\right| dt
\end{aligned}$$

by (2.7)

$$\le p_1 \sum_{-\infty}^\infty \int_{2c2^{N/p_1}}^{2c2^{(N+1)/p_1}} t^{p_1-1} \left|\left\{ \left(\sum_{-\infty}^\infty (2^{ks_0}|D_k(f)|)^{q_0}\right)^{1/q_0} > \frac{1}{2}c2^{N(1/p_0-1/p_1)}t\right\}\right| dt$$

by (2.8)

$$\le p_1 \sum_{-\infty}^\infty \int_{2c2^{N/p_1}}^{2c2^{(N+1)/p_1}} t^{p_1-1} \left|\left\{ \left(\sum_{-\infty}^\infty (2^{ks_0}|D_k(f)|)^{q_0}\right)^{1/q_0} > ct^{p_1/p_0}\right\}\right| dt$$

since $t \approx 2^{N/p_1}$

$$
\begin{aligned}
&\leq p_1 \int_0^\infty t^{p_1-1} \left| \left\{ \left(\sum_{-\infty}^{\infty} (2^{ks_0} |D_k(f)|)^{q_0} \right)^{1/q_0} > c t^{p_1/p_0} \right\} \right| dt \\
&\leq c p_1 \int_0^\infty t^{p_0-1} \left| \left\{ \left(\sum_{-\infty}^{\infty} (2^{ks_0} |D_k(f)|)^{q_0} \right)^{1/q_0} > ct \right\} \right| dt \\
&\leq c \|f\|^{p_0}_{\dot{F}^{s_0,q_0}_{p_0}} \leq c.
\end{aligned}
$$

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