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ON THE DETERMINANT AND THE HOLONOMY OF EQUIVARIANT ELLIPTIC OPERATORS

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ABSTRACT. Let M be a closed oriented smooth manifold, G a compact Lie group consisting of diffeomorphisms of M, $P \to Z$ a principal G-bundle with a connection and D a G-equivariant elliptic operator. Then a locally constant family of elliptic operators and its determinant line bundle over Z are naturally defined by D. Moreover the holonomy of the determinant line bundle is defined by the connection in P. In this note, we give an explicit formula to calculate the holonomy (Theorem 1.4) and give a proof of the Witten holonomy formula (Theorem 1.7) in the special case above.

1. MAIN RESULTS

Let M be a closed oriented smooth manifold, G a compact Lie group consisting of diffeomorphisms of M, $P \to Z$ a principal G-bundle over a smooth manifold Z with a connection and D a G-equivariant elliptic operator. Then, for any $g \in G$, the index of D evaluated at g, $\mathrm{Index}(D, g)$, is defined by

$$\mathrm{Index}(D, g) = \mathrm{tr}(g|_{\ker D}) - \mathrm{tr}(g|_{\mathrm{coker}\, D})$$

and can be calculated by the well-known fixed point formula (cf. [1], [2] or [5]). On the other hand, the determinant of D evaluated at g, $\det(D, g)$, is defined by

$$\det(D, g) = \det(g|_{\ker D}) / \det(g|_{\mathrm{coker}\, D}).$$

Then the next proposition is an immediate consequence of the elementary result of Lemma 1 in Appendix.

Proposition 1.1. *Let $g \in G$ be any element of finite order p. Then the next equality holds*:

$$\det(D, g) = \exp \frac{2\pi i}{p} \sum_{k=1}^{p-1} \frac{1}{1 - e^{-2\pi i k/p}} \{\mathrm{Index}(D) - \mathrm{Index}(D, g^k)\}$$

where $\mathrm{Index}(D) = \mathrm{Index}(D, 1)$ is the numerical index of D.

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Remark 1.2. The homomorphism $\det(D, \cdot): G \to S^1$ defined by $\det(D, g)$ is determined by its restriction to the dense subset of G which consists of all elements of finite order.

Now we assume that M is a $2n$-dimensional closed Riemannian manifold with a Spin^c-structure and a connection in the associated S^1-bundle of the Spin^c-structure. We assume that G acts on M as isometries and that the action of G preserves the Spin^c-structure and the S^1-connection. Let E be a hermitian vector bundle (or virtual vector bundle) over M with a unitary connection. We assume that the action of G lifts to a connection-preserving unitary action on E. Then we can define the G-equivariant Spin^c-Dirac operator D on M

$$D: \Gamma(S^+ \otimes E) \to \Gamma(S^- \otimes E)$$

(for the definition of the half spinor bundles $S^{\pm}$, see [7, pp. 106–108]) and a line bundle $\det(D)$ over Z by

$$\det(D) = P \times_G ((\wedge^r \ker D)^* \otimes (\wedge^s \operatorname{coker} D))$$

where r, s denote the dimensions of the finite-dimensional (complex) G-modules $\ker D$, $\operatorname{coker} D$. Note that the G-equivariant elliptic operator D naturally defines a locally constant elliptic family $P \times_G D$ parametrized by Z and the determinant line bundle defined by this elliptic family is isomorphic to $\det(D)$ above (see [6, pp. 133–134]). The connection in P naturally defines a connection in $\det(D)$ and we can regard $\det(D)$ as a line bundle with a connection. On the other hand, for any $g \in G$, by considering the mapping torus

$$(1.3) \qquad M_g = M \times [0, 1]/\sim \qquad \text{where} \quad (m, 0) \sim (g(m), 1),$$

we can also define a locally constant family of Dirac operators parametrized by S^1. Here the horizontal subspaces of the fibration $M_g \to S^1$ is given by the [0,1]-directed vectors. Then we can define as in [7] the determinant line bundle $L(g)$ over S^1. Note that, if a horizontal lift $\tilde{\gamma}$ of an oriented loop γ in Z connects any fixed base point b in P with $b \cdot g^{-1}$, it is not difficult to see that $L(g)$ is isomorphic to the restriction of $\det(D)$ to the loop γ as a line bundle with a connection. Now it can be seen that the holonomy of $L(g)$ around S^1, which we denote by $\operatorname{hol}(D, g)$, is equal to $\det(D, g)$ and hence the next theorem follows immediately from Proposition 1.1.

Theorm 1.4. *Let $g \in G$ be any element of finite order p. Then we have*

$$(1.5) \qquad \operatorname{hol}(D, g) = \exp \frac{2\pi i}{p} \sum_{k=1}^{p-1} \frac{1}{1 - e^{-2\pi i k/p}} \{\operatorname{Index}(D) - \operatorname{Index}(D, g^k)\},$$

and hence $\operatorname{hol}(D, g)$ is calculated explicitly by using the Atiyah-Bott-Singer fixed point formula.

Now the tangent bundle of M_g splits as the direct sum of the tangent bundle of M and the trivial real line bundle defined by [0,1]-directed vectors. Hence the Riemannian metric and the Spin^c-structure on M_g are naturally defined by those on M together with the standard metric and the trivial Spin^c-structure on [0,1]. Moreover the associated S^1-bundle of the Spin^c-structure over M_g and its connection are naturally defined by the S^1-bundle over M and the standard globally flat connection in the [0,1]-directed trivial real line bundle. Let S_g be the spinor bundle with respect to the above Spin^c-structure on M_g and E_g the hermitian vector bundle (or virtual vector bundle) over M_g with a unitary connection defined by the mapping torus construction (1.3). Then the Spin^c-Dirac operator

$$A_g : \Gamma(S_g \otimes E_g) \to \Gamma(S_g \otimes E_g)$$

is defined and

$$\xi_g = \frac{1}{2}(\eta_g + \dim \ker A_g)$$

is defined by the eta invariant η_g of A_g. Note that, using the same argument as in [4], we can see that ξ_g modulo integer is continuous in g.

Now we assume that $g \in G$ has a finite order p. Let $X = M \times D^2$, $Y = \partial X = M \times S^1$ be the product Riemannian Spin^c-manifolds with the Spin^c-structures induced from the Spin^c-structure on M and the trivial Spin^c-structures on D^2, S^1. We give the metric $p^2 ds^2$ on S^1 where ds^2 is the standard metric on S^1 and a rotationally symmetric Riemannian metric on D^2 which is the product metric near $\partial D^2 = S^1$. The Levi-Civita connection of this metric defines the connection in the associated S^1-bundles over D^2, S^1. Then we can define the actions of $\mathbb{Z}_p = \langle g \rangle$ on X and on $\partial X = Y$ as follows:

$$g \cdot (m\,,\, re^{i\theta}) = (g(m)\,,\, re^{i\theta + 2\pi i/p})$$

for $(m, re^{i\theta}) \in X = M \times D^2$; $0 \le r \le \frac{p}{2\pi}$, $0 \le \theta \le 2\pi$. Note that $\mathbb{Z}_p$ acts freely on Y and $Y/\mathbb{Z}_p$ is equal to M_g. Let $q_X : X = M \times D^2 \to M$, $q_Y : Y = M \times S^1 \to M$ be projections, and let E_X, E_Y denote the hermitian vector bundles (or virtual vector bundles) $q_X^* E = E \times D^2$, $q_Y^* E = E \times S^1$ on X, Y provided with naturally induced unitary connections. Let

$$B : \Gamma(S_X^+ \otimes E_X) \to \Gamma(S_X^- \otimes E_X),$$
$$A : \Gamma(S_Y \otimes E_Y) \to \Gamma(S_Y \otimes E_Y)$$

be the $\mathbb{Z}_p$-equivariant Spin^c-Dirac operators on X, Y where $S_X^\pm$, S_Y are the spinor bundles over X, Y with respect to the Spin^c-structures on X, Y respectively. Then it is clear that the spinor bundle S_g is equal to the spinor bundle with respect to the Spin^c-structure on M_g induced from the $\mathbb{Z}_p$-invariant Spin^c-structure on Y. Moreover it is also clear that E_g is equal to the quotient $E_Y/\mathbb{Z}_p$ and that the Spin^c-Dirac operator A_g on M_g is equal to the quotient $A/\mathbb{Z}_p$. Here we have the following:

Proposition 1.6. *Let $g \in G$ be any element of finite order p. Then we have*

$$\text{the right-hand side of } (1.5) = (-1)^{\mathrm{Index}(D)} e^{-2\pi i \xi_g}.$$

Proof. For any $h \in \mathbb{Z}_p$, let $\eta_Y(h)$ denote the eta invariant of A evaluated at h (cf. [4]). Then it follows from the same arguments as in [8] that

$$\xi_g = \frac{1}{2}(\eta_g + \dim \ker A_g) = \frac{1}{p} \sum_{k=1}^{p} \left(\frac{1}{2}\eta_Y(g^k) + \frac{1}{2}\mathrm{tr}(g^k|_{\ker A}) \right).$$

On the other hand, it follows from Theorem 1.2 in [8] that

$$\frac{1}{2}\eta_Y(g^k) + \frac{1}{2}\mathrm{tr}(g^k|_{\ker A}) + \mathrm{Index}(B\,,\, g^k)$$

is equal to the integral

$$\int_X \mathrm{ch}(E_X)\exp\frac{\mathrm{c}_1(S,X)}{2}\widehat{A}(X)$$

if $k=p$, and is equal to the summation of certain characteristic numbers $\mathfrak{A}[N]$

$$\sum_{N\subset\Omega(X)}\mathfrak{A}[N]$$

if $k\neq p$, where $\mathrm{Index}(B,g^k)$ is the g^k-index (i.e., the index evaluated at g^k) of B with the global boundary condition considered in Theorem (3.10) in [3], $\mathrm{ch}(E_X)$ is the Chern character form of E_X, $\mathrm{c}_1(S,X)$ is the first Chern form of the associated S^1-bundle of the Spin^c-structure on X with respect to the S^1-connection, $\widehat{A}(X)$ is the total $\widehat{A}$-form of TX and $\Omega(X)$ is the fixed point set of the g^k-action ($k\neq p$) on X consisting of closed connected submanifolds N. Now it is easy to see that the fixed point set $\Omega(X)$ coincides with the fixed point set $\Omega(M)$ of the g^k-action on $M=M\times\{0\}\subset M\times D^2=X$ and the normal bundles $\nu(N,X)$ of N in X is isomorphic to the direct sum of the normal bundles $\nu(N,M)$ of N in M and the trivial bundles $N\times\mathbb{R}^2$. Here g acts on $N\times\mathbb{R}^2$ via the $2\pi/p$-rotation of the fiber $\mathbb{R}^2$. Hence, considering the fixed point formula (cf. [5]), we can see that the quantity $\sum_{N\subset\Omega(X)}\mathfrak{A}[N]$ is related to the index of the operator D on M as follows:

$$\sum_{N\subset\Omega(X)}\mathfrak{A}[N]=\frac{1}{1-e^{-2\pi ik/p}}\mathrm{Index}(D,g^k).$$

On the other hand, it is clear that

$$\mathrm{c}_1(S,X)=q_X^*\,\mathrm{c}_1(S,M)+q_D^*\,\mathrm{c}_1(D^2)$$

where $\mathrm{c}_1(S,M)$ is the first Chern form of the associated S^1-bundle of the Spin^c-structure on M with respect to the S^1-connection, $q_D\colon X=M\times D^2\to D^2$ is the projection and $\mathrm{c}_1(D^2)$ is the first Chern form of D^2 with respect to the S^1-connection which is rotationally symmetric and is product near the boundary. Moreover, since

$$\mathrm{ch}(E_X)=q_X^*\,\mathrm{ch}(E),\quad \widehat{A}(X)=q_X^*\,\widehat{A}(M)$$

and

$$\int_{D^2}\exp\frac{\mathrm{c}_1(D^2)}{2}=\int_{D^2}\frac{\mathrm{c}_1(D^2)}{2}=\frac{1}{2},$$

it follows that

$$\int_X \mathrm{ch}(E_X)\exp\frac{\mathrm{c}_1(S,X)}{2}\widehat{A}(X)=\frac{1}{2}\int_M \mathrm{ch}(E)\exp\frac{\mathrm{c}_1(S,M)}{2}\widehat{A}(M)=\frac{1}{2}\mathrm{Index}(D).$$

Hence we can deduce the following equality.

$$\xi_g=\frac{1}{p}\sum_{k=1}^{p-1}\frac{1}{1-e^{-2\pi ik/p}}\mathrm{Index}(D,g^k)+\frac{1}{2p}\mathrm{Index}(D)-\frac{1}{p}\sum_{k=1}^{p}\mathrm{Index}(B,g^k).$$

Now it follows from Lemma 2 in Appendix that

$$\frac{1}{p}\sum_{k=1}^{p}\mathrm{Index}(B,g^k)=0\quad\mathrm{mod.}\mathbb{Z}$$

and from Lemma 3 in Appendix that

$$\frac{1}{2p}\mathrm{Index}(D) = \frac{1}{2}\mathrm{Index}(D) - \frac{1}{p}\sum_{k=1}^{p-1}\frac{1}{1-e^{-2\pi ik/p}}\mathrm{Index}(D).$$

Thus we can conclude that

$$e^{-2\pi i\xi_g} = (-1)^{\mathrm{Index}(D)}\exp\frac{2\pi i}{p}\sum_{k=1}^{p-1}\frac{1}{1-e^{-2\pi ik/p}}\{\mathrm{Index}(D) - \mathrm{Index}(D, g^k)\}.$$

This completes the proof.

Since both $\mathrm{hol}(D, g)(= \det(D, g))$ and $e^{-2\pi i\xi_g}$ are continuous in g, it follows from Remark 1.2, Theorem 1.4 and Proposition 1.6 that

Theorem 1.7 (cf. [7]). *The next equality holds*:

$$\mathrm{hol}(D, g) = (-1)^{\mathrm{Index}(D)}e^{-2\pi i\xi_g}$$

for any $g \in G$.

2. An example

Let M be the non-singular hypersurface of degree $p \geq 2$ in $\mathbb{CP}^{n+1}$ defined by

$$z_0^p + z_1^p + \cdots + z_{n+1}^p = 0$$

where $[z_0 : z_1 : \cdots : z_{n+1}]$ is the homogeneous coordinate of $\mathbb{CP}^{n+1}$. Then the action

$$g\cdot[z_0 : z_1 : \cdots : z_{n+1}] = [e^{2\pi i/p}z_0 : z_1 : \cdots : z_{n+1}]$$

defines an action of $\mathbb{Z}_p = \langle g\rangle$ on M and the fixed point set of this action is the non-singular hypersurface of degree p in $\mathbb{CP}^n = \{z_0 = 0\} \subset \mathbb{CP}^{n+1}$ defined by

$$z_1^p + z_2^p + \cdots + z_{n+1}^p = 0.$$

Let D be the Dolbeault operator on M which is a $\mathbb{Z}_p$-equivariant elliptic operator. Then it follows from the Atiyah-Bott-Singer fixed point formula (see, for example, [9]) that $\mathrm{Index}(D)$ is equal to the x^n-coefficient of

$$\left(\frac{x}{1-e^{-x}}\right)^{n+2}\left(\frac{1-e^{-px}}{px}\right) \in \mathbb{C}[[x]]$$

multiplied by p and that $\mathrm{Index}(D, g^k)$ is equal to the x^{n-1}-coefficient of

$$\left(\frac{x}{1-e^{-x}}\right)^{n+1}\left(\frac{1-e^{-px}}{px}\right)\frac{1}{1-e^{-x}e^{-2\pi ik/p}} \in \mathbb{C}[[x]]$$

multiplied by p.

Now, for example, consider the case of $n = 2, 3$. Then we can obtain Tables 1 and 2 only from direct computations using Theorem 1.4 and the fixed point formula above.

TABLE 1

$\boxed{n=2}$

p	3	4	5	6	7	8	9	10	11	12	13	14	15	16
log(hol)	0	$\frac{1}{4}$	0	$\frac{3}{6}$	0	$\frac{6}{8}$	0	0	0	$\frac{3}{12}$	0	$\frac{7}{14}$	0	$\frac{12}{16}$

TABLE 2

$\boxed{n=3}$

p	3	4	5	6	7	8	9	10	11	12	13	14	15	16
log(hol)	0	0	$\frac{4}{5}$	0	0	0	0	$\frac{8}{10}$	0	0	0	0	$\frac{12}{15}$	0

where log(hol) denotes $\frac{1}{2\pi i}\log \operatorname{hol}(D, g) \mod \mathbb{Z}$.

Remark 2.1. If $c_1(M) > 0$ (namely, $p \leq n+1$), it follows from the Kodaira vanishing theorem that $\operatorname{coker} D = \{0\}$ and that $\ker D$ is equal to the 1-dimensional space of constant functions on M on which $\mathbb{Z}_p$ acts trivially. Therefore it immediately follows that $\operatorname{hol}(D, g) = \det(D, g) = 1$ and hence that $\log(\text{hol}) = 0$. This can also be proved from direct calculations similar as above using the Atiyah-Bott-Singer fixed point formula.

APPENDIX

Lemma 1. *Let A be an $(N \times N)$-matrix which satisfies $A^p = E$ for some positive integer p where E denotes the unit matrix. Then the next equality holds*:

$$\det(A) = \exp \frac{2\pi i}{p} \sum_{k=1}^{p-1} \frac{1}{1 - e^{-2\pi i k/p}} \{N - \operatorname{tr}(A^k)\}.$$

Proof. Let $e^{2\pi i \lambda_j/p}$ $(1 \leq j \leq N)$ be the eigenvalues of A where λ_j's are integers such that $1 \leq \lambda_j \leq p$. Then the equality of the lemma is equivalent to the next equality:

$$\lambda_1 + \cdots + \lambda_N = \sum_{k=1}^{p-1} \frac{1}{1 - e^{-2\pi i k/p}} \sum_{j=1}^{N} \left(1 - e^{2\pi i \lambda_j k/p}\right) \mod p.$$

Therefore it suffices to show that

$$\sum_{k=1}^{p-1} \frac{1 - e^{2\pi i k \lambda/p}}{1 - e^{-2\pi i k/p}} = \lambda \mod p \tag{1}$$

for any integer λ such that $1 \leq \lambda \leq p$. Here the left-hand side of (1) is equal to $-\sum_{k=1}^{p-1} \sum_{\nu=1}^{\lambda} e^{2\pi i k \nu/p}$ and hence (1) follows from the equality

$$\sum_{k=1}^{p-1} e^{2\pi i \nu k/p} = -1 \mod p$$

for any integer ν. $\square$

Lemma 2. *Let* V *be any finite-dimensional* $\mathbb{Z}_p$*-module and* $g \in \mathbb{Z}_p$. *Then we have*

$$\sum_{k=1}^{p} \mathrm{tr}(g^k|_V) = 0 \quad \mathrm{mod.}p\,.$$

Proof. This lemma follows from the equality

$$\sum_{k=1}^{p} \alpha^k = 0 \quad \mathrm{mod.}p$$

for any complex number α such that $\alpha^p = 1$. □

Lemma 3. *The next equality holds*:

$$\frac{1}{2p} = \frac{1}{2} - \frac{1}{p}\sum_{k=1}^{p-1} \frac{1}{1 - e^{-2\pi ik/p}}\,.$$

Proof. This lemma follows from the equality

$$\sum_{k=1}^{p-1} \frac{1}{1 - e^{-2\pi ik/p}} = \frac{p-1}{2}. \quad \square$$

References

1. M. F. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes* I, Ann. of Math. (2) **86** (1967), 374–407.
2. ——, *A Lefschetz fixed point formula for elliptic complexes* II. *Applications*, Ann. of Math. (2) **88** (1968), 451–491.
3. M. F. Atiyah, V. K. Patodi, and I. M. Singer, *Spectral asymmetry and Riemannian geometry* I, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69.
4. ——, *Spectral asymmetry and Riemannian geometry* II, Math. Proc. Cambridge Philos. Soc. **78** (1975), 405–432.
5. M. F. Atiyah and I. M. Singer, *The index of elliptic operators* III, Ann. of Math. (2) **87** (1968), 546–604.
6. ——, *The index of elliptic operators* IV, Ann. of Math. (2) **92** (1970), 119–138.
7. J. M. Bismut and D. S. Freed, *The analysis of elliptic families* II, Comm. Math. Phys. **107** (1986), 103–163.
8. H. Donnelly, *Eta invariants for G-spaces*, Indiana Univ. Math. J. **27** (1978), 889–918.
9. A. Hattori, *Spin^c-structures and S^1-actions*, Invent. Math. **48** (1978), 7–31.

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