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A MEASURE WITH A LARGE SET OF TANGENT MEASURES

TOBY O'NEIL

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ABSTRACT. There exists a Borel regular, finite, non-zero measure μ on $\mathbb{R}^d$ such that for μ-a.e. x the set of tangent measures of μ at x consists of all non-zero, Borel regular, locally finite measures on $\mathbb{R}^d$.

INTRODUCTION

Tangent measures were introduced in [1] in order to investigate the local behaviour of measures. The main advantage of tangent measures is that they often possess more regularity than the original measure and thus a wider range of analytical techniques may be used upon them. The object of this note is to show that in general this does not necessarily hold.

Let $\mathcal{M}$ be the set of all Borel regular, locally finite, measures on $\mathbb{R}^d$. A sequence (μ_k) of measures in $\mathcal{M}$ converges to μ in $\mathcal{M}$ if $\int f\,d\mu_k \to \int f\,d\mu$ as $k \to \infty$ for all continuous functions f with bounded support. This is equivalent to requiring that $\int g\,d\mu_k \to \int g\,d\mu$ as $k \to \infty$ for all nonnegative functions g with Lipschitz constant less than or equal to 1 and bounded support.

$\mathcal{M}$ together with this notion of convergence is metrisable and the resulting space is both complete and separable. For further information about these results see either [1] or [2].

For $\mu \in \mathcal{M}$, $x \in \mathbb{R}^d$ and $r > 0$ define for $E \subset \mathbb{R}^d$

$$\mu_{x,r}(E) := \mu(x + rE) := \mu(\{x + re : e \in E\}).$$

Suppose that $\mu \in \mathcal{M}$ and $x \in \mathbb{R}^d$. A measure $\nu \in \mathcal{M}$ is said to be a tangent measure of μ at x if ν is not the zero measure (denoted by $\mathbf{0}$) and there exist sequences $r_k \searrow 0$ and $c_k > 0$ such that

$$c_k \mu_{x,r_k} \to \nu \quad \text{as } k \to \infty.$$

The set of all tangent measures to μ at x will be denoted by $\operatorname{Tan}(\mu, x)$.

$\operatorname{Tan}(\mu, x)$ has the following properties:

1. $c\nu \in \operatorname{Tan}(\mu, x)$ whenever $\nu \in \operatorname{Tan}(\mu, x)$ and $c > 0$.
2. $\nu_{0,r} \in \operatorname{Tan}(\mu, x)$ whenever $\nu \in \operatorname{Tan}(\mu, x)$ and $r > 0$.

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3. $\operatorname{Tan}(\mu, x)$ is a closed set with respect to the space of all non-zero, Borel regular, locally finite measures.

As a direct consequence we have:

Lemma 1. *If $\mathscr{N} \subset \operatorname{Tan}(\mu, x)$, then $\bigcup_{r,s>0} r\mathscr{N}_{0,s} \subset \operatorname{Tan}(\mu, x)$ where $r\mathscr{N}_{0,s} := \{r\nu_{0,s} : \nu \in \mathscr{N}\}$.*

Lemma 2. *If $\mathscr{N} \subset \operatorname{Tan}(\mu, x)$ and $\mathscr{N}$ is dense in $\mathscr{M}$, then $\operatorname{Tan}(\mu, x) = \mathscr{M}\setminus\{\mathbf{0}\}$.*

Construction of the measure

Theorem 3. *There exists a non-zero measure $\mu \in \mathscr{M}$ such that for μ-a.e. x, $\operatorname{Tan}(\mu, x) = \mathscr{M}\setminus\{\mathbf{0}\}$.*

Proof. First let us define for $x \in \mathbb{R}^d$ the Dirac measure at x as follows

$$\delta_x(E) := \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Additionally let $\mathbb{Q}^+$ denote the positive rationals and $\mathbb{Q}^d$ denote the rational d-tuples, that is, d-tuples whose coordinates are all rational numbers. We have that

$$\mathscr{S} = \left\{ \alpha_0\delta_0 + \sum_{i=1}^{n-1} \alpha_i \delta_{x_i} : n \in \{2, 3, \dots\}, \alpha_i \in \mathbb{Q}^+, x_i \in \mathbb{Q}^d, |x_i| \le 1 \text{ for } i \in \{0, \dots, n-1\} \text{ and } \sum_{i=0}^{n-1} \alpha_i = 1 \text{ and } i \neq j \Rightarrow x_i \neq x_j \right\}$$

is a countable set, and if $\nu \in \mathscr{S}$, then it is a probability measure with support in $B(0, 1)$ (the closed ball with centre the origin and radius 1). Moreover

$$\bigcup_{p,q\in\mathbb{Q}^+} p\mathscr{S}_{0,q}$$

is a countable set which is dense in $\mathscr{M}$. Thus by Lemmas 1 and 2 it suffices to construct a measure μ such that $\operatorname{Tan}(\mu, x) \supset \mathscr{S}$ for μ-a.e. x.

Let $(\mu_k)_{k=1}^{\infty}$ be a sequence of elements of $\mathscr{S}$ such that every element of $\mathscr{S}$ occurs infinitely many times in this sequence. Thus each μ_k is of the form

$$\mu_k = \alpha(k, 0)\delta_0 + \sum_{i=1}^{n_k-1} \alpha(k, i)\delta_{x(k,i)}$$

where the $\alpha(k, i)$, $x(k, i)$ fulfill the appropriate conditions of $\mathscr{S}$ (in particular $x(k, 0) = 0$). For each μ_k define

$$\sigma_k = \min_{0\le i,j\le n-1} \{|x(k, i) - x(k, j)| : i \neq j\}.$$

From this define an increasing sequence of real numbers (r_k) by setting $r_1 = 8$ and choosing $r_{k+1} > 8^{k+2} r_k/\sigma_k$.

Let $\Sigma := \prod_{k=1}^{\infty}\{0, \dots, n_k - 1\}$, and let P be the probability measure on Σ obtained by setting

$$P(\eta|_j) := \prod_{k=1}^{j} \alpha_{k,\eta_k}$$

where $\eta|_j := (\eta_1, \ldots, \eta_j) \times \prod_{k=j+1}^{\infty}\{0, \ldots, n_k - 1\}$. Define $\pi: \Sigma \to B(0, 1)$ by

$$\pi(\eta) := \sum_{k=1}^{\infty}(r_k)^{-1}x(k, \eta_k).$$

Notice that π is a well-defined 1-1 map. Set $\mu := \pi_{\#}P$, that is, for $E \subset \mathbb{R}^d$ define

$$\mu(E) := P[\pi^{-1}(E)].$$

I claim that μ is our required measure. The Borel regularity of μ follows from the continuity of the mapping π with respect to the product topology on Σ.

Lemma 4. *For a given $\nu \in \mathcal{S}$, let $(v_i)_{i=1}^{\infty}$ be a strictly increasing sequence such that $\mu_{v_i} = \nu$ for all i. Let*

$$V_\nu = \{\eta \in \Sigma\colon \eta_{\nu(i)} = 0 \ i.o.\}.$$

Then $P(V_\nu) = 1$ and so $\mu[\pi(V_\nu)] = 1$.

Proof. We have that for all i

$$P(\eta_{\nu(i)} = 0) = \alpha(v(i), 0) = \alpha > 0;$$

therefore $\sum P(\eta_{\nu(i)} = 0) = \infty$ and so, by the Borel-Cantelli lemma and independence, the lemma follows. □

Let $V = \bigcap_{\nu \in S} V_\nu$. Then as $\mathcal{S}$ is countable $P(V) = 1$ and so $\mu[\pi(V)] = 1$. For $x \in \pi(\Sigma)$ define $x_i := x(i, [\pi^{-1}(x)]_i)$ and so $x = \sum_{i=1}^{\infty} x_i/r_i$. Let $\overline{x} \in \pi(V)$, and let $\overline{\eta}$ be the associated element of V. Fix $\nu \in \mathcal{S}$, and define $(v_i)_{i=1}^{\infty}$ as in the lemma (so $\mu_{v(i)} = \nu$). Then, as $\overline{\eta} \in V$, there is an infinite set $N \subset \bigcup_{i=1}^{\infty}\{v_i\}$ such that for all $k \in N$, $\overline{x}_k = 0$ and $\mu_k = \nu$.

We wish to show that $\nu \in \operatorname{Tan}(\mu, \overline{x})$. So we need to find sequences $c_j > 0$ and $s_j \searrow 0$ such that $c_j\mu_{\overline{x}, s_j} \to \nu$ as $j \to \infty$.

Let $s_j = 1/r_{k(j)}$ where $k(j)$ is the jth element of N and so $s_j \searrow 0$.

Define

$$c_j = [\mu\{x \in \pi(\Sigma)\colon x_i = \overline{x}_i \text{ for } i = 1, \ldots, k(j) - 1\}]^{-1}.$$

By the equivalence from the introduction, $\phi_k \to \phi$ iff $\int g\, d\phi_k \to \int g\, d\phi$ where $\operatorname{Lip}(g) \leq 1$ and $\operatorname{spt}(g)$ is bounded and g is nonnegative. So fix such a g and suppose $\operatorname{spt}(g) \subset B(0, R)$ for some $R \geq 0$.

Choose $J \in \mathbf{N}$ such that $\frac{27}{28}8^{k(J)} > R$. For $j \geq J$ we have (letting $k := k(j)$)

$$\begin{aligned}\int_{\mathbf{R}^d} g\, d(c_j\mu_{\overline{x}, s_j}) &= c_j \int_{\mathbf{R}^d} g(r_{k(j)}(x - \overline{x}))\, d\mu(x)\\ &= c_j \int_{\pi(\Sigma)} g\left(r_k \sum_{k=1}^{\infty} \frac{x_i - \overline{x}_i}{r_i}\right) d\mu(x).\end{aligned}$$

Let us consider $r_k \sum_{i=1}^{\infty} \frac{x_i - \overline{x}_i}{r_i}$ in more detail. There are two possible cases:

Case 1. $x_i = \overline{x}_i$ for $i = 1, \ldots, k - 1$. Then since

$$r_k \sum_{i=1}^{\infty} \frac{x_i - \overline{x}_i}{r_i} = x_k - \overline{x}_k + r_k \sum_{i=k+1}^{\infty} \frac{x_i - \overline{x}_i}{r_i}$$

and as

$$\left| r_k \sum_{i=k+1}^{\infty} \frac{x_i - \overline{x}_i}{r_i} \right| \le \frac{2}{7} 8^{-k},$$

we have (as $\overline{x}_k = 0$)

$$\left| x_k - r_k \sum_{i=1}^{\infty} \frac{x_i - \overline{x}_i}{r_i} \right| \le \frac{2}{7} 8^{-k}.$$

Case 2. There exists $u \in \{1, \dots, k-1\}$ such that $x_i = \overline{x}_i$ for $i = 1, \dots, u-1$ but $x_u \neq \overline{x}_u$. Thus

$$\sum_{i=1}^{\infty} \frac{x_i - \overline{x}_i}{r_i} = \frac{x_u - \overline{x}_u}{r_u} + \sum_{i=u+1}^{\infty} \frac{x_i - \overline{x}_i}{r_i}$$

and both

$$\left| \sum_{i=u+1}^{\infty} \frac{x_i - \overline{x}_i}{r_i} \right| \le \frac{\sigma_u}{7 r_u} 8^{-k} \quad \text{and} \quad \left| \frac{x_u - \overline{x}_u}{r_u} \right| \ge \frac{\sigma_u}{r_u};$$

therefore

$$\left| r_k \sum_{i=1}^{\infty} \frac{x_i - \overline{x}_i}{r_i} \right| \ge \frac{27}{28} \frac{r_k}{r_u} \sigma_u > \frac{27}{28} 8^k > R.$$

Thus in Case 2, $g[r_k(x - \overline{x})] = 0$ and so

$$c_j \int_{\pi(\Sigma)} g[r_k(x - \overline{x})] \, d\mu(x) = c_j \int_X g[r_k(x - \overline{x})] \, d\mu(x)$$

where $X = \{x \in \pi(\Sigma) \colon x_i = \overline{x}_i \text{ for } i = 1, \dots, k-1\}$. Notice that $c_j = [\mu(X)]^{-1}$. As $\mathrm{Lip}(g) \le 1$, we have by Case 1 that for $x \in X$

$$|g[r_k(x - \overline{x})] - g(x_k)| \le \frac{2}{7} 8^{-k}.$$

Thus integrating over X and multiplying by c_j gives

$$\left| c_j \int_{\pi(\Sigma)} g[r_k(x - \overline{x})] \, d\mu(x) - \frac{1}{\mu(X)} \int_X g(x_k) \, d\mu(x) \right| \le \frac{2}{7} 8^{-k},$$

but by independence,

$$\int_X g(x_k) \, d\mu(x) = \mu(X) \int_{\pi(\Sigma)} g(x_k) \, d\mu(x) = \mu(X) \int_{\mathbb{R}^d} g(x) \, d\nu(x)$$

and so the theorem follows. □

References

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Department of Mathematics, University College London, Gower Street, London, WC1E 6BT, England

Current address: Mathematical Institute, North Haugh, St. Andrews, Fife, KY16 9SS, United Kingdom

E-mail address: `tco@st-and.ac.uk`