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A NOTE ON THE THUE INEQUALITY

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ABSTRACT. We show that for an arbitrary binary form $F(X, Y)$, there is no non-trivial lower bound for the area of the region $|F(x, y)| \leq 1$ which depends only on the discriminant of F.

1. INTRODUCTION

Let $F(X, Y) = a_0X^n + a_1X^{n-1}Y + \cdots + a_nY^n$ be a binary form with complex coefficients and let A_F denote the area of the region $|F(x, y)| \leq 1$ in the real affine plane. Let D_F denote the discriminant of F. If F has the factorization $\prod_{i=1}^{n}(\alpha_iX - \beta_iY)$ with $\alpha_i, \beta_i \in \mathbb{C}$ (every binary form with complex coefficients has such a factorization), then $D_F = \prod_{i<j}(\alpha_i\beta_j - \alpha_j\beta_i)^2$.

In [4], we showed that if F has degree $n \geq 3$ and discriminant $D_F \neq 0$, then

$$|D_F|^{1/n(n-1)}A_F \leq 3B\left(\frac{1}{3}, \frac{1}{3}\right) \tag{1}$$

where $B(\frac{1}{3}, \frac{1}{3})$ denotes the Beta function with arguments of $1/3$. From this, we deduced that $A_F \leq 3B\left(\frac{1}{3}, \frac{1}{3}\right)$ when F is such a form with integer coefficients (since $D_F \in \mathbb{Z}$ in this case). The bound $3B(\frac{1}{3}, \frac{1}{3})$, which has approximate numerical value 15.8997, is attained in both these inequalities when $F(X, Y) = XY(X - Y)$.

As noted in [4], this result has significant implications in the enumeration of solutions of Thue inequalities. A *Thue inequality* is a Diophantine inequality of the type $|F(x, y)| \leq h$ where F is a binary form with integer coefficients which is irreducible and has degree $n \geq 3$, and h is a non-zero integer. In a seminal paper of 1909, Thue [11] showed that for such a form F, the equation $F(x, y) = h$ (hence the inequality $|F(x, y)| \leq h$) has only a finite number of solutions in integers x and y. In 1934, Mahler [7] showed that the number $N_F(h)$ of solutions of the Thue inequality $|F(x, y)| \leq h$ and the area $A_Fh^{2/n}$ of this region are related in the following way:

$$|N_F(h) - A_Fh^{2/n}| \leq ch^{1/(n-1)}$$

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where c is a number (left unspecified by Mahler) which depends only on F. Estimates for $N_F(h)$ involving $A_F h^{2/n}$ have also been given by Mueller and Schmidt [9]. Recently, Thunder [12] applied (1) to prove a special case of a conjecture of Schmidt.

In view of (1) and its application, Julia Mueller asked the author whether it is possible to obtain a positive absolute lower bound for $|D_F|^{1/n(n-1)}A_F$. In this note, we will show that there is no such lower bound for $|D_F|^{1/n(n-1)}A_F$ even over the class of forms with integer coefficients. In fact, we will show that in general, an upper bound on the size of either one of the quantities A_F or $|D_F|$ does not yield a positive lower bound on the other.

2. Main results

If F is a binary *cubic* form with *real* coefficients and non-zero discriminant, then $|D_F|^{1/6}A_F \geq \sqrt{3}B(\frac{1}{3}, \frac{1}{3})$; in fact,

$$|D_F|^{1/6}A_F = \begin{cases} 3B\left(\frac{1}{3}, \frac{1}{3}\right), & \text{if } D_F > 0, \\ \sqrt{3}B\left(\frac{1}{3}, \frac{1}{3}\right), & \text{if } D_F < 0 \end{cases}$$

(see [2]). However, for $n \geq 4$, there is no positive absolute lower bound for $|D_F|^{1/n(n-1)}A_F$. Put $M_n = \max |D_F|^{1/n(n-1)}A_F$ where the maximum is taken over all forms of degree n with complex coefficients and $D_F \neq 0$.

Proposition. *Fix $n \geq 4$. Over the class of forms of degree n with real coefficients, the quantity $|D_F|^{1/n(n-1)}A_F$ assumes all real values between 0 and M_n.*

Proof. The quantity

$$|D_F|^{1/n(n-1)}A_F = |D_F|^{1/n(n-1)} \int_{-\infty}^{\infty} \frac{dv}{|F(1, v)|^{2/n}}$$

is continuous in the roots of $F(1, y)$, and is maximized by a form with real coefficients for which $F(1, y)$ has n distinct real roots (Theorem 3, [4]). Hence, we need only exhibit a sequence of forms G_N such that $|D_{G_N}|^{1/n(n-1)}A_{G_N} \to 0$.

In Theorem 3 of [3], we showed that if $F(X, Y) = \prod_{i=1}^{n}(\alpha_i X - \beta_i Y)$, where the α's and β's are real numbers such that $\alpha_1/\beta_1 < \cdots < \alpha_n/\beta_n$, then

$$|D_F|^{1/n(n-1)}A_F = \sum_{k=1}^{n} D_{F_k}^{1/n(n-1)} \int_0^1 \frac{dt}{F_k(1, t)^{2/n}}$$

where

$$F_k(X, Y) = XY(X - Y)(X - u_1^{(k)}Y)\cdots(X - u_{n-3}^{(k)}Y)$$

and

$$u_j^{(k)} = \frac{(\alpha_{k+1}\beta_k - \alpha_k\beta_{k+1})(\alpha_{k-1}\beta_{k-1-j} - \alpha_{k-1-j}\beta_{k-1})}{(\alpha_{k+1}\beta_{k-1} - \alpha_{k-1}\beta_{k+1})(\alpha_k\beta_{k-1-j} - \alpha_{k-1-j}\beta_k)}$$

for $1 \leq j \leq n-3$, $1 \leq k \leq n$, with the subscripts taken modulo n. In particular, $0 < u_1^{(k)} < u_2^{(k)} < \cdots < u_{n-3}^{(k)} < 1$, for each k.

Let G_N be the form defined by putting $\alpha_i = N^i$ and $\beta_i = 1$ ($N > 1$). Then, with these values of α_i and β_i, we have $u_j^{(1)} \to 0$, $u_j^{(n)} \to 1$, and for $k \neq 1, n$

$$u_j^{(k)} \to \begin{cases} 0, & \text{if } k - j - 1 > 0, \\ 1, & \text{if } k - j - 1 \leq 0 \end{cases}$$

as $N \to \infty$; for example,

$$u_j^{(n)} = \frac{(N - N^n)(N^{n-1} - N^{n-1-j})}{(N - N^{n-1})(N^n - N^{n-1-j})}$$

and the dominant term in both the numerator and denominator is N^{2n-1}. We will show that, in general,

$$\begin{aligned} &D_{F_k}^{1/n(n-1)} \int_0^1 \frac{dt}{F_k(1, t)^{2/n}} \\ &\quad < B\left(1 - \frac{2}{n}, \frac{1}{n(n-1)}\right) \prod_{j=1}^{n-3} \left(u_j^{(k)}(1 - u_j^{(k)})\right)^{1/n(n-1)(n-3)}. \end{aligned} \tag{2}$$

It will then follow that $|D_{G_N}|^{1/n(n-1)} A_{G_N} \to 0$ as $N \to \infty$.

To simplify notation, let us suppose that $k = 1$ and write $u_j = u_j^{(1)}$. Using the estimates, $u_j < 1$, and $|u_l - u_j| < 1 - u_j$ for $j < l$, we have

$$D_{F_1}^{1/n(n-1)} < \prod_{j=1}^{n-3} \left(u_j^{1/(n-3)}(1 - u_j)^{2(n-j-2)}\right)^{1/n(n-1)}.$$

Put $\varepsilon = 1/n(n-1)(n-3)$ and $q = \prod_{j=1}^{n-3} u_j^{\varepsilon}(1 - u_j)^{\varepsilon}$. Then, using the estimates $1 - u_j < 1 - u_j t$ and $(1 - u_j t)^{-1} < (1 - t)^{-1}$ we have

$$\begin{aligned} D_{F_1}^{1/n(n-1)} \int_0^1 \frac{dt}{F_1(1, t)^{2/n}} &< q \int_0^1 \frac{\prod_{j=1}^{n-3}(1 - u_j t)^{2((n-j-2)/n(n-1)) - \varepsilon} dt}{t^{2/n}(1 - t)^{2/n} \prod_{j=1}^{n-3}(1 - u_j t)^{2/n}} \\ &= q \int_0^1 \frac{\prod_{j=1}^{n-3}(1 - u_j t)^{-2((j+1)/n(n-1)) - \varepsilon} dt}{t^{2/n}(1 - t)^{2/n}} \\ &\le q \int_0^1 \frac{dt}{t^{2/n}(1 - t)^{1 - 1/n(n-1)}} \\ &= qB\left(1 - \frac{2}{n}, \frac{1}{n(n-1)}\right). \end{aligned}$$

This establishes (2).

Now from (2), it is clear that $D_{F_k}^{1/n(n-1)} \int_0^1 F_k(1, t)^{-2/n} dt \to 0$ if either $u_1^{(k)} \to 0$ or $u_{n-3}^{(k)} \to 1$. Consequently, $|D_{G_N}|^{1/n(n-1)} A_{G_N} \to 0$ as $N \to \infty$.

Corollary. *Fix $n \ge 4$. Over the class of forms of degree n with integer coefficients, the quantity $|D_F|^{1/n(n-1)} A_F$ assumes arbitrarily small positive values.*

Proof. Since the forms G_N of the preceding proof have integer coefficients when N is a positive integer, it is clear that there is no positive absolute lower bound for $|D_F|^{1/n(n-1)} A_F$ over the class of forms with integer coefficients even though the set of values $\{|D_F|^{1/n(n-1)} A_F\}$ is countable in this case.

Remark. For the cubic forms $F^{(t)}(X, Y) = (X^2 + Y^2)(Y - itX)$, $t \in (0, 1)$, we have $A_{F^{(t)}} = \int_0^1 z^{-1/2}(1 - z)^{-1/2}(1 - (1 - t^2)z)^{-1/3} dz \le B\left(\frac{1}{2}, \frac{1}{6}\right)$ and $|D_{F^{(t)}}|^{1/6} =$

$2^{1/3}(1-t^2)^{1/3}$; so $|D_{F^{(t)}}|^{1/6}A_{F^{(t)}} \to 0$ as $t \to 1$. Hence $|D_F|^{1/6}A_F$ assumes all values between 0 and $3B\left(\frac{1}{3}, \frac{1}{3}\right)$ over the cubic forms with *complex* coefficients.

3. Related examples

It is clear from (1) that if either of the quantities $|D_F|^{1/n(n-1)}$ or A_F is large, then the other must be small since a lower bound for one implies an upper bound for the other. It is natural to ask whether the converse situation holds; that is, whether an *upper* bound for one implies a *lower* bound for the other. In this section, we will give examples to show that this is not the case. We will also show that there are forms F with integer coefficients and $D_F = 0$ for which A_F is arbitrarily large or arbitrarily small.

For any complex number k, $A_{kF} = |k|^{-2/n}A_F$ and $|D_{kF}| = |k|^{2(n-1)}|D_F|$. Hence, by appropriately choosing k, we can ensure that either $A_F < 1$ or $|D_F| < 1$ as desired.

Example 3.1. Consider the forms

$$F^{(t)}(X, Y) = k(X+itY)(X+i(t+1)Y)\cdots(X+i(t+n-1)Y), \quad t \in \mathbb{R}.$$

Notice that $|D_{F^{(t)}}|^{1/n(n-1)} = |k|^{2/n}\left((n-1)!(n-2)!\cdots 2!1!\right)^{2/n(n-1)}$ which is independent of t, and that $A_{F^{(t)}} < 4|k|^{-2/n}t^{-1}$ since the region $|F^{(t)}(x, y)| \le 1$ is contained in the region $|x| \le |k|^{-1/n}$, $|y| \le |k|^{-1/n}t^{-1}$. Given $\varepsilon > 0$, we can ensure that $|D_{F^{(t)}}|^{1/n(n-1)} < \varepsilon$ for all t by appropriately choosing k; then, letting $t \to \infty$, we have $A_{F^{(t)}} \to 0$ while $|D_{F^{(t)}}|^{1/n(n-1)} < \varepsilon$. Hence, an upper bound for $|D_F|$ does not imply a lower bound for A_F.

Example 3.2. Consider the forms

$$F^{(t)}(X, Y) = k(X+itY)(X+iY)(X+2iY)\cdots(X+(n-1)iY), \quad t \in (0, 1).$$

In this case,

$$\begin{aligned} A_{F^{(t)}} &= |k|^{-2/n}\int_{-\infty}^{\infty} \frac{du}{|(u^2+t^2)(u^2+1)(u^2+2^2)\cdots(u^2+(n-1)^2)|^{1/n}} \\ &\le |k|^{-2/n}\int_{-\infty}^{\infty} \frac{du}{|u^2(u^2+1)(u^2+2^2)\cdots(u^2+(n-1)^2)|^{1/n}} \\ &\le |k|^{-2/n}\frac{3B\left(\frac{1}{3}, \frac{1}{3}\right)}{((n-1)!(n-2)!\cdots 1!)^{2/n(n-1)}} \end{aligned}$$

which is independent of t; however, $|D_{F^{(t)}}| \to 0$ as $t \to 1$. Hence, an upper bound for A_F does not imply a lower bound for $|D_F|$.

Example 3.3. Consider the binomial forms

$$F(X, Y) = kX^{(n-1)/2}Y^{(n-1)/2}(X-Y)$$

where n is an odd integer greater than three and k is a positive integer. In this case, $D_F = 0$, $A_F < \infty$, and

$$\begin{aligned} A_F &= k^{-2/n}\int_{-\infty}^{\infty} \frac{dv}{|v^{(n-1)/2}(1-v)|^{2/n}} \\ &> k^{-2/n}\int_0^1 \frac{dv}{v^{(n-1)/n}(1-v)^{2/n}} \\ &= k^{-2/n}B\left(\frac{1}{n}, 1-\frac{2}{n}\right). \end{aligned}$$

Notice that A_F can be made large by fixing k and choosing n sufficiently large; on the other hand, A_F can be made small by fixing n and choosing k sufficiently large. Hence, when D_F is zero, A_F can be arbitrarily large or arbitrarily small even if we assume that $F(X, Y) \in \mathbb{Z}[X, Y]$.

Hence we have proved the following result.

Proposition. *There do not exist positive numbers* C, α *for which*

$$A_F \geq \frac{C}{|D_F|^\alpha}$$

for all forms F *of degree* n *with* $D_F \neq 0$. *In particular, an upper bound for one of the quantities* A_F, $|D_F|$ *does not imply a lower bound for the other. Moreover, when* $D_F = 0$, A_F *can be arbitrarily large or arbitrarily small even if we assume that* $F(X, Y) \in \mathbb{Z}[X, Y]$.

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