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# FIXED POINTS VIA "BIASED MAPS"

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ABSTRACT. A generalization of compatible maps called "biased maps" is introduced and used to prove fixed point theorems for Meir-Keeler type contractions involving four maps. Extensions of known results are thereby obtained. In particular, a theorem by Kang and Rhoades is generalized.

## 1. INTRODUCTION

Self-maps $A$ and $S$ of a metric space $(X, d)$ are said to be compatible ([5]) iff $d(SAx_n, ASx_n) \to 0$ whenever $\{x_n\}$ is a sequence in $X$ such that $Ax_n, Sx_n \to t \in X$. Compatible mappings were introduced in [5] as a generalization of commuting mappings and have been useful as a tool for obtaining more comprehensive fixed point theorems (see, e.g., [1]–[8], [10]–[16]) and in the study of periodic points [9]. Now we introduce the concept of biased maps by softening the restrictions imposed by compatibility. The result is an appreciable generalization of compatible maps which, as we shall see, proves useful in the "fixed point" arena.

**Definition 1.1.** Let $A$ and $S$ be self-maps of a metric space $(X, d)$. The pair $\{A, S\}$ is *S-biased* iff whenever $\{x_n\}$ is a sequence in $X$ and $Ax_n, Sx_n \to t \in X$, then

$$(*) \quad \alpha d(SAx_n, Sx_n) \leq \alpha d(ASx_n, Ax_n) \quad \text{if } \alpha = \liminf \text{ and if } \alpha = \limsup .$$

Of course, if the inequality in $(*)$ holds with $\alpha = \lim_n$ (which fact presupposes that the indicated limit exists), then $\liminf = \limsup = \lim_n$ and $(*)$ is satisfied. We shall frequently use this fact. The following example shows why we could not restrict $\alpha$ to "$\lim_n$" if the bias concept is to generalize compatibility.

(In this paper we shall use $N$, $Q$, $I_r$, and $I$ to denote the positive integers, the rational numbers, the irrational numbers, and $[0, 1]$, respectively.)

**Example 1.1.** Let $X = I$, and define $A, S : X \to X$ by $Ax = Sx = 1 - x$ for $x \in [0, \frac{1}{2}]$, $Ax = Sx = 0$ for $x \in Q \cap (\frac{1}{2}, 1]$, and $Ax = Sx = 1$ for $x \in I_r \cap (\frac{1}{2}, 1]$. Let $x_{2n} = \frac{1}{2n}$ and $x_{2n-1} = \frac{\sqrt{2}}{2n+1}$ for $n \in N$. Then $Sx_k = 1 - x_k \to 1$ as $k \to \infty$, $SSx_{2n} = 0$, $SSx_{2n-1} = 1$, and therefore $\lim_k d(SSx_k, Sx_k)$ does

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not exist although $\lim_k d(SSx_k, SSx_k) = 0$; in fact, the pair $\{S, S\}$ is trivially compatible for any function $S$.

We remind the reader that $\liminf x_n = \sup\{\overline{x}_n : n \in N \& \overline{x}_n = \inf_{k \geq n} x_k\}$ (for $\limsup$, switch sup and inf), and that if $\alpha = \liminf$ or $\limsup$, $\alpha x_n \leq \alpha y_n$ when $x_n \leq y_n + z_n$ for $n \in N$ and $z_n \to 0$ as $n \to \infty$. Also be assured that the "biased" map concept arises naturally in the context of contractive or relatively nonexpansive ([7]) maps. See Proposition 2.1 below.

*Remark* 1.1. If the pair $\{A, S\}$ is compatible, then it is both $S$- and $A$-biased. For

$$d(SAx_n, Sx_n) \leq d(SAx_n, ASx_n) + d(ASx_n, Ax_n) + d(Ax_n, Sx_n) \quad \text{for } n \in N;$$

therefore, $\alpha d(SAx_n, Sx_n) \leq 0 + \alpha d(ASx_n, Ax_n) + 0$ if $Ax_n, Sx_n \to t \in X$, $\{A, S\}$ is a compatible pair, and $\alpha$ is either $\liminf$ or $\limsup$. Thus $\{A, S\}$ is $S$-biased. Similarly, by interchanging $A$ and $S$ in the above, we conclude that $\{A, S\}$ is $A$-biased if the pair is compatible. On the other hand, consider the following.

**Example 1.2.** Define $A, S : [0, 1] \to [0, 1]$ by $Ax = 1 - 2x$ and $Sx = 2x$ for $x \in [0, \frac{1}{2}]$, and $Ax = 0$, $Sx = 1$ for $x \in (\frac{1}{2}, 1]$. Then, by using Proposition 1.1 below, it is easy to show that $\{A, S\}$ is both $A$- and $S$-biased but not compatible. (Note that both $A$ and $S$ are continuous and $[0, 1]$ is compact, so that both $A$ and $S$ are proper maps; i.e., $A^{-1}(M)$ is compact if $M$ is.)

The next result is the analogue to Theorem 2.2 in [8] for compatible maps.

**Proposition 1.1.** *Let $A$ and $S$ be self-maps of a metric space $(X, d)$.*

(a) *If the pair $\{A, S\}$ is $S$-biased and $Ap = Sp$, then $d(SAp, Sp) \leq d(ASp, Ap)$.*

(b) *If $A$ and $S$ are continuous and one of $A$ or $S$ is proper, then $\{A, S\}$ is $S$-biased iff $Ap = Sp$ implies that $d(SAp, Sp) \leq d(ASp, Ap)$.*

*Proof.* To see that (a) holds, suppose that $Ap = Sp$. Let $x_n = p$ for $n \in N$, so $Ax_n = Sx_n \to Ap = Sp$. Then $d(SAp, Sp) = \lim_n d(SAx_n, Sx_n) \leq \lim_n d(ASx_n, Ax_n) = d(ASp, Ap)$ as desired, since $\{A, S\}$ is $S$-biased.

Of course, the necessity portion of (b) follows from (a). To see that the condition given in (b) is sufficient to ensure that $\{A, S\}$ is $S$-biased, suppose that $\{x_n\}$ is a sequence such that $Ax_n, Sx_n \to t \in X$ and that $S$ is proper. Then $M = \{Sx_n, n \in N\} \cup \{t\}$ is compact and therefore $S^{-1}(M)$ is compact. But then the sequence $\{x_n\}$ in $S^{-1}(M)$ has a subsequence $\{x_{k_n}\}$ which converges to a point $p$, and therefore $\{Ax_{k_n}\}$, $\{Sx_{k_n}\}$ converge to $Ap$ and $Sp$, respectively, since $A$ and $S$ are continuous. Then $Ap = Sp = t$ by "uniqueness of limits", so that $d(SAp, Sp) \leq d(ASp, Ap)$ by hypothesis. But then, since $Ax_n \to Ap = t$ and $Sx_n \to Sp$, $SAx_n \to SAp$ and $ASx_n \to ASp$ because $A$ and $S$ are continuous. We thus have $\lim_n d(SAx_n, Sx_n) \leq \lim_n d(ASx_n, Ax_n)$, as desired. $\square$

In Example 1.2 the pair $\{A, S\}$ was both $A$-biased and $S$-biased. Of course, this need not be the case. Consider:

**Example 1.3.** Let $I = [0, 1]$ with the absolute value metric. Define $A, S : I \to I$ by $A(x) = (x - \frac{1}{2})^2$ and $S(x) = 2A(x)$ for $x \in I$. Then $A$ and $S$ are certainly proper since both are continuous and $I$ is compact. We thus appeal to Proposition 1.1. Now $Ax = Sx$ iff $x = \frac{1}{2}$. Since $A(\frac{1}{2}) = S(\frac{1}{2}) = 0$, $SA(\frac{1}{2}) = S(0) = \frac{1}{2}$ and $AS(\frac{1}{2}) = A(0) = \frac{1}{4}$. Thus $|AS(\frac{1}{2}) - A(\frac{1}{2})| = \frac{1}{4}$ and $|SA(\frac{1}{2}) - S(\frac{1}{2})| = \frac{1}{2}$, so by Proposition 1.1 the pair $\{A, S\}$ is $A$-biased and not $S$-biased. Consequently, Remark 1.1 tells us that $\{A, S\}$ is not compatible. For future reference, note that $|Ax - Ay| = \frac{1}{2}|Sx - Sy|$ for $x, y \in I$.

## 2. $(\varepsilon, \delta)$-CONTRACTIONS FOR FOUR MAPS

Meir-Keeler contractions for four maps were introduced in [5] and called $(\varepsilon, \delta)$-contractions. To expedite the ensuing discussion of theory and results, we extend the $(\varepsilon, \delta)$ concept as follows.

**Definition 2.1.** A pair of self-maps $A$ and $B$ of a metric space $(X, d)$ are $(\varepsilon, \delta)$-$S$, $T(p)$-contractions relative to maps $S, T : X \to X$ iff $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$, and there exist functions $p : XxX \to [0, \infty)$ and $\delta : (0, \infty) \to (0, \infty)$ such that $\delta(\varepsilon) > \varepsilon$ for all $\varepsilon$, and for $x, y \in X$:

(i) $\varepsilon \le p(x, y) < \delta(\varepsilon)$ implies that $d(Ax, By) < \varepsilon$.

We shall refer to $(\varepsilon, \delta)(p)$-contractions as $(m)$ contractions if

$$p(x, y) = m(x, y) = \max\left\{d(Sx, Ty), \frac{1}{2}(d(Sx, By) + d(Ax, Ty))\right\},$$

and as $(M)$ contractions if

$$p(x, y) = M(x, y) = \max\left\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), \frac{1}{2}(d(Sx, By) + d(Ax, Ty))\right\}.$$

*Remark* 2.1. Thus, suppose $A$ and $B$ are $(\varepsilon, \delta)$-$S$, $T(p)$ contractions with $p(x, y) = m(x, y)$ or $M(x, y)$. Then $Ax = By$ when $p(x, y) = 0$. Consequently, if $Ax \ne By$, $p(x, y) \ne 0$ and (i) in Definition 2.1 therefore implies that $d(Ax, By) < p(x, y)$. So in general, $d(Ax, By) \le p(x, y)$ for all $x, y \in X$.

The following proposition tells us that biased maps arise quite naturally. In particular, relatively nonexpansive maps [7] and thus $(\varepsilon, \delta)$-contractions induce "bias".

**Proposition 2.1.** *Let $A, B, S$, and $T$ be self-maps of a metric space $(X, d)$ such that $A(X) \subseteq T(X)$ and $d(Ax, By) \le d(Sx, Ty)$ for $x, y \in X$. If $S$ is continuous, the pair $\{A, S\}$ is $A$-biased.*

*Proof*. For suppose $Ax_n, Sx_n \to t(\in X)$. Since $A(X) \subseteq T(X)$, for each $n \in N$ $\exists y_n \in X$ such that $Ty_n = Ax_n$. Then $d(Ty_n, By_n) = d(Ax_n, By_n) \le d(Sx_n, Ty_n) \to 0$, so $By_n \to t$. We thus have $Ax_n, By_n, Sx_n, Ty_n \to t$. Now

$$d(ASx_n, Ax_n) \le d(ASx_n, By_n) + d(By_n, Ax_n) \quad \text{for } n \in N,$$

so the continuity of $S$ implies

$$\alpha\, d(ASx_n\,,\, Ax_n) \le \alpha\, d(ASx_n\,,\, By_n) \le \lim_n d(SSx_n\,,\, Ty_n) = \lim_n d(SAx_n\,,\, Sx_n)\,,$$

whether $\alpha = \liminf$ or $\limsup$; i.e., $\{A\,,\, S\}$ is $A$-biased. □

On the other hand, Example 1.3 tells us that even though $A = B$ and $S = T$, and both $A$ and $S$ are continuous in Proposition 2.1, the pair $\{A\,,\, S\}$ need not be $S$-biased.

The following result on $(\varepsilon\,,\, \delta)$-contractions will prove useful.

**Proposition 2.2.** *Let $S$ and $T$ be self-maps of a metric space $(X\,,\, d)$, and let $A$ and $B$ be $(\varepsilon\,,\, \delta)$-$S\,,\, T(p)$-contractions of $(X\,,\, d)$ with $\delta$ lower-semicontinuous. If $\{x_n\}$ and $\{y_n\}$ are sequences in $X$ such that $\lim_n p(x_n\,,\, y_n) = \varepsilon > 0$ and $\limsup d(Ax_n\,,\, By_n) = r \in R$, then $r < \varepsilon$.*

*Proof.* Since $\delta(\varepsilon) > \varepsilon$ and $\delta$ is a lower-semicontinuous function, there is a neighborhood $N_\varepsilon$ of $\varepsilon$ such that $\delta(t) > \varepsilon$ for $t \in N_\varepsilon$. We can therefore choose $t_0 \in N_\varepsilon$ such that $0 < t_0 < \varepsilon < \delta(t_0)$. Since $p(x_n\,,\, y_n) \to \varepsilon$, there exists $m \in N$ such that $p(x_n\,,\, y_n) \in (t_0\,,\, \delta(t_0))$ for $n \ge m$. Then, by (i) in Definition 2.1, $d(Ax_n\,,\, By_n) \le t_0$ for $n \ge m$; i.e., $\limsup d(Ax_n\,,\, By_n) = r \le t_0 < \varepsilon$. □

## 3. Main results

Proposition 1.1 prompts the following convenient definition.

**Definition 3.1.** Let $A$ and $S$ be self-maps of a metric space $(X\,,\, d)$. The pair $\{A\,,\, S\}$ is *weakly $S$-biased* iff $Ap = Sp$ implies $d(SAp\,,\, Sp) \le d(ASp\,,\, Ap)$.

Of course, if $\{A\,,\, S\}$ is $S$-biased, it is weakly $S$-biased by Proposition 1.1(a).

**Lemma 3.1.** *Let $A\,,\, B\,,\, S$, and $T$ be self-maps of a metric space $(X\,,\, d)$. Suppose that*

$$(*)\qquad \begin{gathered} Ax \ne By \quad \textit{implies} \\ d(Ax\,,\, By) < m(x\,,\, y) = \max\left\{d(Sx\,,\, Ty)\,,\, \frac{1}{2}(d(Sx\,,\, By) + d(Ax\,,\, Ty))\right\}. \end{gathered}$$

*If there exist $u\,,\, v\,,\, p \in X$ such that $p = Au = Su = Bv = Tv$ and $\{A\,,\, S\}$ is weakly $S$-biased ($\{B\,,\, T\}$ is weakly $T$-biased), then $p = Ap = Sp$ ($p = Bp = Tp$).*

*Proof.* Suppose that $\{A\,,\, S\}$ is weakly $S$-biased. Since $p = Au = Su$, we have $d(SAu\,,\, Su) \le d(ASu\,,\, Au)$; i.e., (1) $d(Sp\,,\, p) \le d(Ap\,,\, p)$. We assert that $Ap = p$, and hence $p = Sp$ by (1). For if $Ap \ne p$, then $Ap \ne Bv$ by hypothesis, and $(*)$ therefore implies that $d(Ap\,,\, p) = d(Ap\,,\, Bv) < m(p\,,\, v) = \max\{d(Sp\,,\, Tv)\,,\, \frac{1}{2}(d(Sp\,,\, Bv) + d(Ap\,,\, Tv))\} = \max\{d(Sp\,,\, p)\,,\, \frac{1}{2}(d(Sp\,,\, p) + d(Ap\,,\, p))\} \le d(Ap\,,\, p)$ by (1). But we then have the contradiction, $d(Ap\,,\, p) < d(Ap\,,\, p)$. The proof that $p = Bp = Tp$ when $\{B\,,\, T\}$ is weakly biased is analogous. □

The proof of the following result uses the fact that any $(m)$ contraction is an $(M)$ contraction.

**Theorem 3.1.** *Let $S$ and $T$ be self-maps of a complete metric space $(X, d)$. Suppose that $A$ and $B$ are $(\varepsilon, \delta)$-$S$, $T(m)$-contractions and that the pair $\{A, S\}$ is $S$-biased and $\{B, T\}$ is $T$-biased. If one of $A$, $B$, $S$, or $T$ is continuous and $\delta$ is lower semicontinuous, then there is a unique point $p \in X$ such that $p = Ap = Bp = Sp = Tp$.*

*Proof.* Let $x_0 \in X$, and let $\{y_n\}$ be defined inductively by $y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$ for $n \in N$. Since $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$, the $x_i$ can be so chosen. As is known (see, e.g., [16], [11]) and not difficult to prove, the sequence $\{y_n\}$ thus defined is Cauchy. Since $X$ is complete, $\exists p \in X$ such that $y_n \to p$. In particular,

$$Ax_{2n},\ Sx_{2n},\ Bx_{2n-1},\ Tx_{2n-1} \to p. \tag{3.1}$$

We first use (3.1) to show that for any sequence $\{v_n\}$ in $X$ and $n \in N$,

(3.2)
(i) $d(Av_n, Bx_{2n-1}) \leq d(Sv_n, Tx_{2n-1}) + \beta_n$,
(ii) $d(Ax_{2n}, Bv_n) \leq d(Sx_{2n}, Tv_n) + \gamma_n$ where $\beta_n, \gamma_n \to 0$ as $n \to \infty$.

To prove (i), note that by definition of $(m)$ contractions,

$$\begin{aligned} d(Av_n, Bx_{2n-1}) &\leq m(v_n, x_{2n-1}) \\ &= \max\left\{d(Sv_n, Tx_{2n-1}), \frac{1}{2}(d(Av_n, Tx_{2n-1}) + d(Sv_n, Bx_{2n-1}))\right\}, \end{aligned}$$

so $d(Av_n, Bx_{2n-1}) \leq d(Sv_n, Tx_{2n-1})$ or $d(Av_n, Bx_{2n-1}) \leq \frac{1}{2}d(Av_n, Tx_{2n-1}) + \frac{1}{2}d(Sv_n, Bx_{2n-1})$ for $n \in N$. The first inequality satisfies (i) with $\beta_n = 0$, so we need consider only the second inequality. But the second inequality and the triangle inequality imply:

$$\begin{aligned} 2d(Av_n, Bx_{2n-1}) \leq\ & d(Av_n, Bx_{2n-1}) + d(Bx_{2n-1}, Tx_{2n-1}) \\ & + d(Sv_n, Tx_{2n-1}) + d(Tx_{2n-1}, Bx_{2n-1}), \end{aligned}$$

which yields: $d(Av_n, Bx_{2n-1}) \leq d(Sv_n, Tx_{2n-1}) + 2d(Bx_{2n-1}, Tx_{2n-1})$. This last inequality produces (i), since (3.1) implies $\beta_n = 2d(Bx_{2n-1}, Tx_{2n-1}) \to 0$.

The proof of (3.2)(ii) follows similarly with $\gamma_n = 2(Ax_{2n}, Sx_{2n})$.

Now assume that one of $S$ or $T$, say $S$, is continuous. Then $SSx_{2n}, SAx_{2n} \to Sp$ by (3.1). We assert that $Sp = p$. For suppose that $d(Sp, p) = \varepsilon > 0$. Then (3.1) implies

$$\begin{aligned} \varepsilon = d(Sp, p) &= \lim_n d(SSx_{2n}, Tx_{2n-1}) \\ &= \lim_n d(SSx_{2n}, Bx_{2n-1}) = \lim_n d(SAx_{2n}, Sx_{2n}). \end{aligned} \tag{3.3}$$

Since $\{A, S\}$ is $S$-biased (see Definition 1.1),

$$\varepsilon = \lim_n d(SAx_{2n}, Sx_{2n}) = \alpha\, d(SAx_{2n}, Sx_{2n}) \leq \alpha\, d(ASx_{2n}, Ax_{2n}).$$

Now $d(ASx_{2n}, Ax_{2n}) \leq d(ASx_{2n}, Bx_{2n-1}) + d(Bx_{2n-1}, Ax_{2n})$ for $n \in N$. Therefore,

$$\varepsilon \leq \alpha\, d(ASx_{2n}, Ax_{2n}) \leq \alpha\, d(ASx_{2n}, Bx_{2n-1}), \tag{3.4}$$

since $d(Bx_{2n-1}, Ax_{2n}) \to 0$.

But $d(ASx_{2n}, Bx_{2n-1}) \le d(SSx_{2n}, Tx_{2n-1}) + \beta_n$ by (3.2)(i), so

$$\begin{aligned}\limsup d(ASx_{2n}, Bx_{2n-1}) &\le \limsup d(SSx_{2n}, Tx_{2n-1}) \\ &= \lim_n d(SSx_{2n}, Tx_{2n-1}).\end{aligned}$$

Then (3.3), (3.4), and the preceding inequality imply that

$$\begin{aligned}0 < \varepsilon &= \lim_n d(ASx_{2n}, Bx_{2n-1}) = \lim_n d(SSx_{2n}, Tx_{2n-1}) \\ &= \lim_n d(SSx_{2n}, Bx_{2n-1}) = \lim_n d(ASx_{2n}, Tx_{2n-1}),\end{aligned}$$

the last equality following from (3.1).

But

$$\begin{aligned}&m(Sx_{2n}, x_{2n-1}) \\ &\quad = \max\{d(SSx_{2n}, Tx_{2n-1}), \frac{1}{2}(d(ASx_{2n}, Tx_{2n-1}) + d(SSx_{2n}, Bx_{2n-1}))\},\end{aligned}$$

so

$$0 < \varepsilon = \lim_n m(Sx_{2n}, x_{2n-1}) = \lim_n d(ASx_{2n}, Bx_{2n-1}),$$

contradicting Proposition 2.2.

Thus, $Sp = p$. But then Remark 2.1 and (3.2) imply that $d(Ap, p) = \lim_n d(Ap, Bx_{2n-1}) \le \lim_n (d(Sp, Tx_{2n-1}) + \beta_n) = d(Sp, p) = 0$. We therefore have $Sp = Ap = p$. But $A(X) \subseteq T(X)$ by hypothesis, so $\exists u \in X$ such that $Tu = Ap = Sp$. Therefore, by Remark 2.1,

$$\begin{aligned}d(Ap, Bu) \le m(p, u) &= \max\{d(Sp, Tu), \tfrac{1}{2}(d(Sp, Bu) + d(Ap, Tu))\} \\ &= \tfrac{1}{2} d(Ap, Bu).\end{aligned}$$

We conclude that $Ap = Bu$, and we have $Bu = Tu = p = Ap = Sp$; consequently, $Bp = Tp = p = Ap = Sp$ by Lemma 3.1.

By symmetry, the argument above applied to $B$ and $T$ yields a common fixed point if $T$ is continuous.

Assume next that $A$ is continuous. Then (3.1) implies that $AAx_{2n}, ASx_{2n} \to Ap$. Suppose that $d(Ap, p) = \varepsilon > 0$. Then

$$0 < \varepsilon = \lim_n d(AAx_{2n}, Bx_{2n-1}) = \lim_n d(ASx_{2n}, Ax_{2n}). \tag{3.5}$$

Since $\{A, S\}$ is $S$-biased,

$$\lim_n d(ASx_{2n}, Ax_{2n}) = \limsup d(ASx_{2n}, Ax_{2n}) \ge \limsup d(SAx_{2n}, Sx_{2n}).$$

But (3.1) implies $\limsup d(SAx_{2n}, Sx_{2n}) = \limsup d(SAx_{2n}, Tx_{2n-1})$, so by (3.5)

$$0 < \varepsilon = \lim_n d(AAx_{2n}, Bx_{2n-1}) \ge \limsup d(SAx_{2n}, Tx_{2n-1}). \tag{3.6}$$

Moreover, by Remark 2.1 and (3.2),

$$\begin{aligned}\liminf d(SAx_{2n}, Tx_{2n-1}) &\ge \liminf d(AAx_{2n}, Bx_{2n-1}) \\ &= \lim_n d(AAx_{2n}, Bx_{2n-1}).\end{aligned}$$

We therefore obtain by (3.6), the preceding inequality, and (3.1)

$$\begin{aligned} 0 < \varepsilon = \lim_n d(AAx_{2n}, Bx_{2n-1}) &= \lim_n d(SAx_{2n}, Tx_{2n-1}) \\ = \lim_n d(AAx_{2n}, Tx_{2n-1}) &= \lim_n d(SAx_{2n}, Bx_{2n-1}), \end{aligned}$$

which implies

$$\lim_n m(Ax_{2n}, x_{2n-1}) = \varepsilon = \lim_n d(AAx_{2n}Bx_{2n-1}),$$

contradicting Proposition 2.2.

We conclude that $Ap = p$. But $A(X) \subseteq T(X)$ implies that $Tv = Ap = p$ for some $v \in X$, so $d(Bv, p) = \lim_n d(Bv, Ax_{2n}) \leq \lim_n(d(Tv, Sx_{2n}) + \gamma_n) = d(p, p) = 0$, by (3.1) and (3.2). Therefore $Bv = p = Tv = Ap$. But $B(X) \subseteq S(X)$, so that $Su = Bv = Tv$ for some $u \in X$ and we obtain as above: $d(Au, Bv) \leq m(u, v) = \frac{1}{2}d(Au, Bv)$. Consequently, $Au = Bv$; therefore $Tv = Bv = p = Su = Au$, which implies $Tp = Bp = p = Ap = Sp$ by Lemma 3.1.

Of course, a completely analogous argument yields a common fixed point if $B$ is assumed to be continuous. We have shown that if one of $A, B, S$, or $T$ is continuous, then $A, B, S$, and $T$ have a common fixed point. The uniqueness of the common fixed point follows immediately from the definition of $(\varepsilon, \delta)$-$S$, $T(m)$-contractions. □

**Corollary 3.1.** *Let $A, B, S, T$ be self-maps of a complete metric space $(X, d)$ such that $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$. Suppose there exists $r \in (0, 1)$ such that*

$$d(Ax, By) \leq rm(x, y) \quad \textit{for } x, y \in X.$$

*If $\{A, S\}$ is $S$-biased and $\{B, T\}$ is $T$-biased, then $A, B, S$, and $T$ have a unique common fixed point, provided one of $A, B, S$, or $T$ is continuous.*

*Proof*. Let $\delta(\varepsilon) = \varepsilon/r$ for $\varepsilon \in (0, \infty)$. Then $\delta : (0, \infty) \to (0, \infty)$, $\delta$ is continuous and therefore certainly lower-semicontinuous, and $\delta(\varepsilon) > \varepsilon$ since $r < 1$. Moreover, $m(x, y) < \delta(\varepsilon) = \varepsilon/r$ implies that $d(Ax, By) \leq rm(x, y) < r(\varepsilon/r) = \varepsilon$, so that $A$ and $B$ are $S$, $T$-$(m)$-contractions. □

Of course, Corollary 3.1 holds if we replace $m(x, y)$ by $d(Sx, Ty)$. However, Example 1.3 shows that even though we were to make that substitution, require that $A = B$, $S = T$, and demand that both $A$ and $S$ be continuous, the conclusion to Corollary 3.1 need not hold if the pair $\{A, S\}$ is not $S$-biased.

The role of "biased" maps in producing fixed points is demonstrated even more dramatically by the next result. If we drop all continuity requirements and the demand that $\delta$ be lower semicontinuous in Theorem 3.1, we can still secure a c.f.p. by merely requiring that one of $A(X)$, $B(X)$, $S(X)$, or $T(X)$ be complete instead of $X$.

**Theorem 3.2.** *Let $S$ and $T$ be self-maps of a metric space $(X, d)$, and let $A, B$ be $(\varepsilon, \delta)$-$S$, $T(m)$-contractions. If one of $A(X)$, $B(X)$, $S(X)$, or $T(X)$ is complete, and the pairs $\{A, S\}$ and $\{B, T\}$ are weakly $S$-biased and weakly $T$-biased respectively, then $A, B, S$, and $T$ have a unique common fixed point.*

*Proof*. As in the proof of Theorem 3.1, there exists a Cauchy sequence $\{y_n\}$ defined by: $y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$ for $n \in N$.

Suppose $T(X)$ is complete. Since $\{y_n\}$ is Cauchy, the subsequence $\{y_{2n-1}\}$ $(\subseteq T(X))$ is Cauchy and therefore converges to a point $p = T(v)$ for some $v \in X$. Then the Cauchy sequence $\{y_n\}$ also converges to $p$, and we have

$$Ax_{2n}, Sx_{2n}, Bx_{2n-1}, Tx_{2n-1} \to p. \tag{3.7}$$

Since $A$ and $B$ are $(\varepsilon, \delta)$-$S$, $T(m)$-contractions, Remark 2.1 and the triangle inequality yield

$$d(p, Bv) \le d(p, Ax_{2n}) + d(Ax_{2n}, Bv) \le d(p, Ax_{2n}) + m(x_{2n}, v);$$

so for $n \in N$,

$$d(p, Bv) \le d(p, Ax_{2n}) + \max\left\{d(Sx_{2n}, Tv), \frac{1}{2}(d(Sx_{2n}, Bv) + d(Ax_{2n}, Tv))\right\}.$$

Then (3.7) implies that $d(p, Bv) \le \frac{1}{2}d(p, Bv)$ as $n \to \infty$, and we infer that $p = Bv = Tv$. But $B(X) \subseteq S(X)$, so there exists $u \in X$ such that $Su = Bv = Tv$. Therefore,

$$d(Au, Bv) \le \max\{d(Su, Tv), \tfrac{1}{2}(d(Su, Bv) + d(Au, Tv))\} = \tfrac{1}{2}d(Au, Bv).$$

Hence, $Au = Bv$, and we have $p = Bv = Tv = Au = Su$. Consequently, our hypothesis, Remark 2.1, and Lemma 3.1 demand that $p = Ap = Bp = Sp = Tp$. That $p$ is the only common fixed point follows from the definition of $(m)$ contractions and Remark 2.1.

In the above we assumed that $T(X)$ was complete. A comparable argument yields (3.7) and hence the conclusion if $S(X)$ is complete. If on the other hand, for example, $A(X)$ is complete, we obtain (3.7) and have $p \in A(X)$. But $A(X) \subseteq T(X)$, so that $p \in T(X)$ and the above argument pertains. □

The following corollary to Theorem 3.2 generalizes the main theorem, Theorem 2.3, of Kang and Rhoades in [13] by eliminating continuity requirements completely and by replacing "compatibility" with "weak bias" and $d(Sx, Ty)$ with $m(x, y)$. (Note that the roles of the pairs $A$, $B$ and $S$, $T$ are reversed in [13].)

**Corollary 3.2.** *Let $A$, $B$, $S$, and $T$ be self-maps of a complete metric space $(X, d)$ with $S$ and $T$ surjective. Suppose that the pair $\{A, S\}$ is weakly $S$-biased and $\{B, T\}$ is weakly $T$-biased. If there is a nondecreasing upper semicontinuous function $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(t) < t$ for all $t > 0$ and*

$$d(Ax, By) \le \varphi(m(x, y)) \quad \text{for } x, y \in X, \tag{3.8}$$

*$A$, $B$, $S$, and $T$ have a unique common fixed point.*

*Proof.* We first show that the pair $A$, $B$ is an $(\varepsilon, \delta)$-$S$, $T$-contraction. Now $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ since $S$ and $T$ are surjections. Since $\varphi$ is u.s.c. and $\varphi(\varepsilon) < \varepsilon$ when $\varepsilon > 0$, for each such $\varepsilon$ $\exists r_\varepsilon > 0$ such that $\varphi(t) < \varepsilon$ for $t \in (\varepsilon - r_\varepsilon, \varepsilon + r_\varepsilon)$. We can therefore define $\delta : (0, \infty) \to (0, \infty)$ by $\delta(\varepsilon) = \sup\{t \in (\varepsilon, \varepsilon + 1) : \varphi(t) < \varepsilon\}$. Clearly, $\delta(\varepsilon) > \varepsilon$ for $\varepsilon > 0$. Moreover, by the above we infer that if $0 < \varepsilon \le t < \delta(\varepsilon)$, the definition of $\delta$ yields $t_0 \in$

$[t, \delta(\varepsilon))$ such that $\varphi(t_0) < \varepsilon$, and hence $\varphi(t) < \varepsilon$, since $\varphi$ is nondecreasing. We conclude that for any $\varepsilon$,

$$0 < \varepsilon \leq t < \delta(\varepsilon) \quad \text{implies} \quad \varphi(t) < \varepsilon.$$

Therefore, if $\varepsilon \leq m(x, y) < \delta(\varepsilon)$, $d(Ax, By) \leq \varphi(m(x, y)) < \varepsilon$ by (3.8). Thus, property (i) in Definition 2.1 is satisfied.

We have shown that the pair $A, B$ is an $(\varepsilon, \delta)$-$S, T$-contraction by Definition 2.1. Moreover, since $T(X) = X$, $T(X)$ is complete. The hypothesis of Theorem 3.2 has been shown to be satisfied, and the unique common fixed point is thereby assured. □

In our consideration of Theorems 3.1 and 3.2, we should ask whether or not these results hold for the more general $(M)$ contractions, where by Definition 2.1,

$$M(x, y) = \max\left\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), \frac{1}{2}(d(Ax, Ty) + d(Sx, By))\right\}.$$

This question merits a reply, since results analogous to Theorem 3.1 for *compatible pairs* $\{A, S\}$ and $\{B, T\}$ which use $M(x, y)$ instead of $m(x, y)$ are in print—e.g., Theorems 8 and 12 in [2], Theorem 3.2 in [3], Theorem 3.1 in [10], or the very general theorem by Rhoades, Park, and Moon in [11] and [16]. The following example shows that although we replace $m(x, y)$ in Theorems 3.1 and 3.2 by $p(x, y) = \max\{d(Ax, Sx), d(By, Ty)\}$ instead of $M(x, y)$ and permit all four functions to be continuous, we need not obtain a common fixed point if we require only *biased pairs* of maps. Note that $m(x, y)$ is obtained from $M(x, y)$ by deleting the $p(x, y)$ terms.

**Example 3.1.** Let $X = I = [0, 1]$ and $d$ the absolute value metric. Define $A, B, S, T : I \to I$ by

$$Ax = Bx = \begin{cases} \frac{1}{2} & \text{if } x \in [0, \frac{1}{4}], \\ 1 - 2x & \text{if } x \in [\frac{1}{4}, \frac{1}{2}], \\ 0 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

and

$$Sx = Tx = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}], \\ 1 & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Clearly, $A$ and $S$ are continuous and $A(X) = [0, \frac{1}{2}] \subseteq S(X) = X$, so that both $A(X)$ and $S(X)$ are complete. Moreover, $A$ and $S$ are proper since $I$ is compact, so we may use Proposition 1.1(b) to show that the pair $\{A, S\}$ is biased. To this end note that $At = St$ iff $t = \frac{1}{4}$. And $A(\frac{1}{4}) = \frac{1}{2} = S(\frac{1}{4})$, so $SA(\frac{1}{4}) = 1$, $AS(\frac{1}{4}) = 0$. Therefore, $|AS(\frac{1}{4}) - A(\frac{1}{4})| = \frac{1}{2} = |SA(\frac{1}{4}) - S(\frac{1}{4})|$, which implies $\{A, S\}$ is both $A$-biased and $S$-biased. But since $SA(\frac{1}{4}) \neq AS(\frac{1}{4})$, $\{A, S\}$ is not compatible ([8, Theorem 2.2]). We now show $|Ax - Ay| \leq \frac{1}{2}|Ay - Sy|$ if $x \leq y$, so $d(Ax, Ay) \leq \frac{1}{2}\max\{d(Ax, Sx), d(Ay, Sy)\}$ certainly holds. (Note: Because of symmetry in $x$ and $y$ in this last inequality, we lose no generality by assuming $x \leq y$.)

Now if $0 \leq x, y \leq \frac{1}{4}$, $|Ax - Ay| = 0 \leq \frac{1}{2}|Ay - Sy|$. If $0 \leq x \leq \frac{1}{4} \leq y \leq \frac{1}{2}$, $|Ax - Ay| = |\frac{1}{2} - (1-2y)| = \frac{1}{2}|4y - 1|$ and $|Ay - Sy| = |(1-2y) - 2y| = |1 - 4y| = 2|Ax - Ay|$. On the other hand, if $\frac{1}{4} \leq x \leq y < \frac{1}{2}$, $|Ax - Ay| = 2(y - x)$, whereas $|Ay - Sy| = 4y - 1 \geq 4y - 4x = 4(y - x)$, since $4x \geq 1$; thus, $|Ax - Ay| \leq \frac{1}{2}|Ay - Sy|$. Finally, if $y \geq \frac{1}{2}$, $|Ay - Sy| = 1$, but for any $x, y : |Ax - Ay| \leq \frac{1}{2}$. We have shown that in any event, $x \leq y$ implies $|Ax - Ay| \leq \frac{1}{2}|Ay - Sy|$. But $A$ and $S$ do not have a common fixed point; neither do $B$ and $T$, since $A = B$ and $S = T$.

The above example clearly demonstrates that the potentially ill-behaved terms in $M(x, y)$ are $d(Ax, Sx)$ and $d(By, Ty)$ when the pairs $\{A, S\}$ and $\{B, T\}$ are not compatible. Consequently, improvements or generalizations of Theorems 3.1 and 3.2 may be difficult to come by in the context of $(M)$ contractions and biased maps. But when the pairs $\{A, S\}$ and $\{B, T\}$ are compatible, Proposition 2.2 in [5] guarantees the desired response from $d(Ax, Sx)$ and $d(By, Ty)$. The interested reader can confirm this by checking the proofs of theorems in [2], [12], and [16], for example.

## 4. Retrospect

By the above, if we require that the pairs $\{A, S\}$ and $\{B, T\}$ be compatible instead of being $S$ and $T$ biased, respectively, Theorem 3.1 is valid for $(M)$ contractions as well as $(m)$ contractions. Therefore, the following "suggests" that Theorem 3.1 may not be a new result.

**Proposition 4.1.** *Let $S$ and $T$ be self-maps of a metric space $(X, d)$, and let $A$ and $B$ be $(\varepsilon, \delta)$-$S$, $T$-$(m)$-contractions with $\delta$ lower semicontinuous. Suppose that the pair $\{A, S\}$ is $S$-biased. If one of $A$ or $S$ is continuous, then the pair $\{A, S\}$ is compatible.*

*Proof.* Suppose $\{x_n\}$ is a sequence in $X$ and $t \in X$ such that $Ax_n, Sx_n \to t$. We can then appeal to the proof of Proposition 2.1 to obtain a sequence $\{y_n\}$ such that $By_n, Ty_n \to t$. If we substitute $x_n$ for $x_{2n}$ and $y_n$ for $x_{2n-1}$ in that portion of the proof of Theorem 3.1 which verifies that $Sp = p$ when $S$ is continuous, we obtain $\lim_n d(ASx_n, By_n) = \lim_n d(SAx_n, Sx_n) = 0$. But

$$d(ASx_n, SAx_n) \leq d(ASx_n, By_n) + d(By_n, Sx_n) + d(Sx_n, SAx_n), \quad \text{for } n \in N,$$

so $d(ASx_n, SAx_n) \to 0$; i.e., $\{A, S\}$ is compatible.

The argument in the instance in which $A$ is continuous is comparable. □

The following example assures us that, in spite of Proposition 4.1, Theorem 3.1 pertains to situations not included by Theorem 2.1 of [6]. We again refer to Remark 2.1 and remind the reader that Proposition 4.1 certainly holds for $(\varepsilon, \delta)$-$S$, $T$-contractions.

**Example 4.1.** Let $X = [0, 1]$. We define maps $A, B, S, T : X \to X$ such that $A(X) = \{\frac{1}{2}\} \subseteq T(X) = \{0\} \cup [\frac{1}{2}, 1]$, $B(X) = \{\frac{1}{2}, \frac{3}{8}\} = S(X)$, and only $A$ is continuous. These facts will be immediately apparent, and we leave them for the reader to confirm. Now define

$$Ax = Bx = Sx = \frac{1}{2} \quad \text{and} \quad Tx = 1 - x \quad \text{if } x \in \left[0, \frac{1}{2}\right],$$

and

$$Ax = \frac{1}{2}, \quad Bx = Sx = \frac{3}{8}, \quad \text{and} \quad Tx = 0 \quad \text{if } x \in \left(\frac{1}{2}, 1\right].$$

First note that $\{A, B\}$ is an $(\varepsilon, \delta)$-$S$, $T$-contraction since it satisfies $|Ax - By| \leq \frac{1}{3}|Sx - Ty|$ for $x, y \in X$. To see this, observe that $|Ax - By| \neq 0$ only when $y > \frac{1}{2}$. Then $|Ax - By| = \frac{1}{8}$; whereas $Sx \geq \frac{3}{8}$ for all $x$, so that $|Sx - Ty| = |Sx - 0| \geq \frac{3}{8}$. Thus, in any event, $3|Ax - By| \leq |Sx - Ty|$. To see that $\{A, S\}$ is compatible, suppose that $Ax_n, Sx_n \to t \in X$. Clearly, $t = \frac{1}{2}$ and $x_n \leq \frac{1}{2}$ for large $n$ since $|Ax - Sx| = \frac{1}{8}$ for $x > \frac{1}{2}$. Then $SAx_n = S(\frac{1}{2}) = \frac{1}{2}$ and $ASx_n = \frac{1}{2}$. Thus $|SAx_n - ASx_n| \to 0$. On the other hand, consider $B$ and $T$. If $Bx_n, Tx_n \to t \in X$, then $t = \frac{1}{2}, x_n \to \frac{1}{2}$, and $x_n \leq \frac{1}{2}$ for large $n$. So $Tx_n \in \{\frac{1}{2}, 1 - x_n\}$, $Bx_n = \frac{1}{2}$, and $TBx_n = \frac{1}{2}$ for all large $n$. Then $|TBx_n - Tx_n| \to |\frac{1}{2} - \frac{1}{2}| = 0$, and $\{B, T\}$ is therefore $T$-biased. On the other hand, if $x_n = \frac{1}{2} - \frac{1}{n}$, e.g., $Tx_n \to \frac{1}{2}^+$ and therefore $|BTx_n - Bx_n| \to |\frac{3}{8} - \frac{1}{2}| = \frac{1}{8}$. Consequently, $\{B, T\}$ is not $B$-biased and thus not compatible.

We conclude by noting that Theorem 3.2 eliminated all continuity requirement on $A$, $B$, $S$, and $T$, and the l.s.c. requirements on $\delta$ imposed in Theorem 3.1, and merely required that one of the range spaces be complete in lieu of $X$ being complete. This prompts the

**Question.** To what extent can the lower semicontinuity hypothesis on $\delta$ be muted in Theorem 3.1?

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