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MUTUALLY COMPLEMENTARY FAMILIES OF T_1 TOPOLOGIES, EQUIVALENCE RELATIONS AND PARTIAL ORDERS

JURIS STEPRĀNS AND STEPHEN WATSON

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ABSTRACT. We examine the maximum sizes of mutually complementary families in the lattice of topologies, the lattice of T_1 topologies, the semi-lattice of partial orders and the lattice of equivalence relations. We show that there is a family of κ many mutually complementary partial orders (and thus T_0 topologies) on κ and, using this family, build another family of κ many mutually T_1 complementary topologies on κ. We obtain κ many mutually complementary equivalence relations on any infinite cardinal κ and thus obtain the simplest proof of a 1971 theorem of Anderson. We show that the maximum size of a mutually T_1 complementary family of topologies on a set of cardinality κ may not be greater than κ unless $\omega < \kappa < 2^c$. We show that it is consistent with and independent of the axioms of set theory that there be $\aleph_2$ many mutually T_1-complementary topologies on ω_1 using the concept of a splitting sequence. We construct small maximal mutually complementary families of equivalence relations.

1. HISTORY AND INTRODUCTION

In 1936, Birkhoff published *On the combination of topologies* in Fundamenta Mathematicae [7]. In this paper, he ordered the family of all topologies on a set by letting $\tau_1 < \tau_2$ if and only if $\tau_1 \subset \tau_2$. He noted that the family of all topologies on a set is a lattice. That is to say, for any two topologies τ and σ on a set, there is a topology $\tau \wedge \sigma$ which is the greatest topology contained in both τ and σ (actually $\tau \wedge \sigma = \tau \cap \sigma$) and there is a topology $\tau \vee \sigma$ which is the least topology which contains both τ and σ. This lattice has a greatest element, the discrete topology, and a smallest element, the indiscrete topology, whose open sets are just the null set and the whole set. In fact, the lattice of all topologies on a set is a complete lattice; that is to say there is a greatest topology contained in each element of a family of topologies and there is a least topology which contains each element of a family of topologies.

A sublattice of this lattice which contains all the Hausdorff spaces is the lattice of T_1 topologies. This is also a complete lattice whose smallest element is the

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cofinite topology (whose open sets are just the null set and those sets whose complements are finite).

We say that topologies τ and σ are complementary if and only if $\tau \wedge \sigma = 0$ and $\tau \vee \sigma = 1$. In 1965, Steiner [16] used a careful analysis of an argument of Gaifman [14] to show that the lattice of all topologies on any set is complemented. On the other hand, there is an elementary example of a T_1 topology which has no complement in the lattice of T_1 topologies. We shall speak of a complement in the lattice of T_1 topologies as a T_1-complement.

Removing the antisymmetric axiom from the list of axioms for the theory of partial orders yields the larger class of preorders. The family of all preorders on a set can also be ordered by letting $\mathbb{P}_0 < \mathbb{P}_1$ if and only if $\mathbb{P}_0 \supset \mathbb{P}_1$ as relations. The family of all preorders on a set is also a lattice. The join of two preorders is just their intersection as relations, while their meet is the transitive closure of their union. In fact, once again, the lattice of all preorders on a set is a complete complemented lattice. The subfamily of partial orders is a semi-lattice. Complementation is quite natural in the lattice of preorders as well as in the semi-lattice of partial orders. Two partial orders (or preorders) are complementary if and only if their intersection as relations is the identity relation and the transitive closure of their union is the improper (largest) relation.

The family of all equivalence relations on a set can also be ordered by letting $\sim_0 < \sim_1$ if and only if $\sim_0 \supset \sim_1$. The family of all equivalence relations on a set is also a complete complemented lattice [15]. In fact, it is a sublattice of the lattice of preorders.

The lattices of topologies, T_1 topologies, preorders and equivalence relations and the semi-lattices of T_0 topologies and partial orders should be studied together. In 1935, Alexandroff and Tucker [2], [1], [17] independently observed a close relationship between topologies and preorders. If $<$ is a preorder on a set X, then the *AT* (*Alexandroff-Tucker*) *topology* on X is that topology obtained by letting $\{y : y \geq x\}$ be open for each $x \in X$. Conversely, if X is a topological space, then the *induced preorder* $\lhd$ is defined by $a \lhd b \Leftrightarrow a \in \overline{b}$ or, equivalently, $a \lhd b \Leftrightarrow (\forall U \in \tau)(a \in U \Rightarrow b \in U)$.

These relationships are inverse [16] in the sense that if $(X, \lhd)$ is a preordered set and τ is the AT topology on X, then the induced preorder on (X, τ) is equal to $\lhd$. Furthermore, if we define a topological space to be an AT space if and only if each point has a smallest neighborhood, then if (X, τ) is an AT space and $\lhd$ is the induced preorder on X, then the AT topology on $(X, \lhd)$ is equal to τ.

These observations mean that preorders 'are' topological spaces and, in fact, partial orders 'are' T_0 topological spaces. Equivalence relations are preorders and thus also topological spaces. Of course, the topology which corresponds to an equivalence relation which is not just the identity relation is not T_0.

We have studied the nature of complementation in these lattices in [20] and [19]; an extensive bibliography can be found in the former.

In this paper, we study the existence of families of topologies, each pair of which are complements. These families of *mutually complementary* topologies turn out to be rather hard to construct. The first construction was of three mutually T_1 complementary topologies on the integers and was accomplished by Anderson and Stewart in 1968 [4]. In 1971, Anderson [3] gave a difficult and interesting construction, for each infinite cardinal κ, of a family of κ many mutually complementary topologies on a set of cardinality κ. In that paper, he also constructed a family of κ many mutually T_1-complementary topologies on a set of cardinality κ. Anderson then turned to the finite case [5], [6] and obtained some nice estimates of the maximum size of a mutually complementary family of topologies on a set of cardinality n. In fact, his topologies in the finite case 'are' equivalence relations. The second author has, with Jason Brown, obtained some results on complementation in the finite case [11], [8] and, in particular, has obtained [10] estimates of the number of mutually complementary partial orders (T_0 topologies) on a finite set.

This completes the short bibliography for the study of the existence of families of mutually complementary structures on a fixed set.

In this paper, we restrict our attention to the infinite case but examine the maximum sizes of mutually complementary families in the lattice of topologies, the lattice of T_1 topologies, the semi-lattice of partial orders and the lattice of equivalence relations.

We observe that Anderson's construction of κ many mutually complementary topologies on κ is best possible. We remark that Anderson's method does not yield mutually complementary families in either the semi-lattice of partial orders, the semi-lattice of T_0 topologies, the lattice of preorders, or the lattice of equivalence relations. We show, by induction, that there is indeed a family of κ many mutually complementary partial orders (and thus T_0 topologies) on κ. Furthermore, we show that the structure of such partial orders can be taken, in some sense, as simple as possible.

We obtain an alternate construction of a family of κ many mutually T_1 complementary topologies on κ by showing that there is a family of κ many mutually complementary partial orders of a certain kind on κ. Furthermore, this construction is direct rather than by induction. We then show that any such family can be modified to be a family of mutually T_1 complementary topologies.

We obtain, by induction, κ many mutually complementary equivalence relations on any infinite cardinal κ. This elegant construction provides what is probably the simplest and least technical proof of Anderson's 1971 result.

We note that our observation in the case of the lattice of all topologies does not remove the possibility that there may be more than κ many mutually T_1 complementary topologies on κ. A subtler argument shows that the maximum size of a mutually T_1 complementary family of topologies on a set of cardinality κ may be greater than κ but only for κ which are uncountable and less than $2^{\mathfrak{c}}$. Nevertheless, we show that it is consistent with and independent of the axioms of set theory that there be $\aleph_2$ many mutually T_1-complementary topologies on ω_1. We leave open, however, the possibility that there may be a construction, without using special axioms of set theory, of $2^{\mathfrak{c}}$ many mutually T_1-complementary topologies on $\mathfrak{c}$.

We use the concept of a *splitting sequence* in the proof of consistency of the existence of more than κ many mutually T_1 complementary topologies on κ. We believe that this concept is of independent interest and ask some questions about the existence of splitting sequences of various kinds.

We show that these mutually complementary families must be constructed with some care by showing that there are maximal families of mutually complementary equivalence relations on κ of any size between three and κ. Indeed, these families are maximal even among preorders.

We conclude the paper with the observation that complementation can arise in a 'finitary' way or an 'infinitary' way. We define this observation in a rigorous way and pose some questions whose investigation we hope will shed further light on the nature of complementation.

2. Lower bounds for partial orders

The complexity of a partial order can be measured in a very crude way by the maximum size of chains. The partial orders we construct in Theorem 1 have no chains of size more than 3. On the other hand, there is no family of even three mutually complementary partial orders with no chains of size 3 (every element must be maximal or minimal and no element can be maximal (or minimal) in two mutually complementary partial orders).

Theorem 1. *If κ is an infinite cardinal, then there are κ mutually complementary partial orders on κ. Furthermore each maximal chain in each of these partial orders has size* 3.

Proof. We use the notation id_β to indicate the identity relation on β (here and in Lemma 3 below). We construct κ many mutually complementary topologies by fixing the index set of the topologies and adding points one at a time. That is, we choose a listing

$$L = \{(\alpha_0(\xi), \alpha_1(\xi), \gamma_0(\xi), \gamma_1(\xi)) : \xi \in \kappa\}$$

of κ^4 in order type κ in which each term appears cofinally and $(\forall \xi \in \kappa)(\forall i \in 2)\alpha_i(\xi), \gamma_i(\xi) < \xi$ and then define a family of partial orders $\{<_\alpha^\beta : \alpha, \beta \in \kappa\}$ and a family of mappings $\{\pi_\alpha^\beta : \alpha, \beta \in \kappa\}$ so that

(1) $<_\alpha^\beta \subset \beta^2$,
(2) $\beta_0 < \beta_1 \Rightarrow <_\alpha^{\beta_0} = <_\alpha^{\beta_1} \cap \beta_0^2$,
(3) $\alpha_0 \neq \alpha_1 \Rightarrow <_{\alpha_0}^\beta \cap <_{\alpha_1}^\beta = \mathrm{id}_\beta$,
(4) $\pi_\alpha^\beta : \beta \to 3$,
(5) $\beta_0 < \beta_1 \Rightarrow \pi_\alpha^{\beta_0} \subset \pi_\alpha^{\beta_1}$,
(6) $\gamma <_\alpha^\beta \delta \Rightarrow$ either $\pi_\alpha^\beta(\gamma) = 0 \wedge \pi_\alpha^\beta(\delta) = 1$ or $\pi_\alpha^\beta(\gamma) = 1 \wedge \pi_\alpha^\beta(\delta) = 2$ or $(\exists \eta \in \beta) : \gamma <_\alpha^\beta \eta <_\alpha^\beta \delta \wedge \pi_\alpha^\beta(\gamma) = 0 \wedge \pi_\alpha^\beta(\eta) = 1 \wedge \pi_\alpha^\beta(\delta) = 2$,
(7) if $\pi_{\alpha_0(\beta)}^\beta(\gamma_0(\beta)) = 1 = \pi_{\alpha_1(\beta)}^\beta(\gamma_1(\beta))$, then $\pi_{\alpha_0(\beta)}^{\beta+1}(\beta) = 0$, $\pi_{\alpha_1(\beta)}^{\beta+1}(\beta) = 2$, $\beta <_{\alpha_0(\beta)}^{\beta+1} \gamma_0(\beta)$ and $\gamma_1(\beta) <_{\alpha_1(\beta)}^{\beta+1} \beta$,
(8) $(\forall \gamma > 0)(\exists! \alpha) : \pi_\alpha^{\gamma+1}(\gamma) = 0$,
(9) $(\forall \gamma > 0)(\exists! \alpha) : \pi_\alpha^{\gamma+1}(\gamma) = 2$,
(10) if $\pi_\alpha^\beta(\gamma) = 0$ then $\exists \gamma'(\pi_\alpha^\beta(\gamma') = 1 \wedge \gamma <_\alpha^\beta \gamma')$, if $\pi_\alpha^\beta(\gamma) = 2$ then $\exists \gamma'(\pi_\alpha^\beta(\gamma') = 1 \wedge \gamma' <_\alpha^\beta \gamma)$.

We accomplish this construction by induction on β only. We start by setting each $\pi_\alpha^1(0) = 1$ and setting each $<_\alpha^1 = \{(0, 0)\}$. Suppose $\{<_\alpha^\nu : \alpha \in \kappa, \nu \in \beta\}$ and a family of mappings $\{\pi_\alpha^\nu : \alpha \in \kappa, \nu \in \beta\}$ have already been defined.

If β is a limit ordinal, then conditions (2) and (5) uniquely define $<_\alpha^\beta$ and π_α^β.

Let us therefore assume that β is the successor of the ordinal β^-. We need only define each $\pi^\beta_\alpha(\beta^-)$ and each $<^\beta_\alpha \cap((\{\beta^-\} \times \beta) \cup (\beta \times \{\beta^-\}))$ and then use (2) and (5).

We consult the listing L for the β^--th 4-tuple

$$(\alpha_0(\beta^-), \alpha_1(\beta^-), \gamma_0(\beta^-), \gamma_1(\beta^-)) = (\alpha_0, \alpha_1, \gamma_0, \gamma_1)$$

and, if $\pi^{\beta^-}_{\alpha_0}(\gamma_0) = 1 = \pi^{\beta^-}_{\alpha_1}(\gamma_1)$, then we define $\pi^\beta_{\alpha_0}(\beta^-) = 0$, $\pi^\beta_{\alpha_1}(\beta^-) = 2$ and all other $\pi^\beta_\alpha(\beta^-) = 1$. We declare $\beta^- <^\beta_{\alpha_0} \gamma_0$ and $\gamma_1 <^\beta_{\alpha_1} \beta^-$ and add no other order relations between β^- and the elements of β^- except those induced by transitivity.

Now we define $<_\alpha = \bigcup\{<^\beta_\alpha : \beta \in \kappa\}$ and $\pi_\alpha = \bigcup\{\pi^\beta_\alpha : \beta \in \kappa\}$.

We claim that $\{<_\alpha : \alpha \in \kappa\}$ is a mutually complementary family of partial orders. Join follows from condition (3). To show meet suppose that U is open in both $<_{\alpha_0}$ and $<_{\alpha_1}$. If U is nonempty, then by condition (10), U must either contain β with $\pi_{\alpha_0}(\beta) = 2$ or else contain δ with $\pi_{\alpha_0}(\delta) = 1$. In the latter case, find β such that $(\alpha_0(\beta), \alpha_1(\beta), \gamma_0(\beta), \gamma_1(\beta)) = (\alpha_1, \alpha_0, 0, \delta)$ with $\beta > \delta$ and, by condition (7), we have $\pi_{\alpha_0}(\beta) = 2$ and $\delta <_{\alpha_0} \beta$ so in either case U contains β with $\pi_{\alpha_0}(\beta) = 2$. By condition (9), $\pi_{\alpha_1}(\beta) = 0$ or $\pi_{\alpha_1}(\beta) = 1$. In the former case, by condition (10), there is $\gamma \in U$ with $\pi_{\alpha_1}(\gamma) = 1$. Thus, in either case, there is $\gamma \in U$ such that $\pi_{\alpha_1}(\gamma) = 1$. If U is proper, then, by condition (10), U must either miss β with $\pi_{\alpha_1}(\beta) = 0$ or else miss δ with $\pi_{\alpha_1}(\delta) = 1$. In the latter case, find β such that $(\alpha_0(\beta), \alpha_1(\beta), \gamma_0(\beta), \gamma_1(\beta)) = (\alpha_1, \alpha_0, \delta, 0)$ with $\beta > \delta$ and, by condition (7), we have $\pi_{\alpha_1}(\beta) = 0$ and $\beta <_{\alpha_1} \delta$. So, in either case, U misses β with $\pi_{\alpha_1}(\beta) = 0$. By condition (8), $\pi_{\alpha_0}(\beta) = 2$ or $\pi_{\alpha_0}(\beta) = 1$. In the former case, by condition (10), there is $\gamma' \notin U$ with $\pi_{\alpha_0}(\gamma') = 1$. Thus, in either case, there is $\gamma' \notin U$ such that $\pi_{\alpha_0}(\gamma') = 1$. Find $\eta \in \kappa$ such that $(\alpha_0(\eta), \alpha_1(\eta), \gamma_0(\eta), \gamma_1(\eta)) = (\alpha_0, \alpha_1, \gamma', \gamma)$ where $\eta > \gamma_0, \gamma_1$. By condition (7), $\eta <_{\alpha_0} \gamma'$ and $\gamma <_{\alpha_1} \eta$. Thus $\eta \notin U$ and $\eta \in U$ which is impossible.

We shall provide a construction of κ many T_1 complementary topologies on a set of cardinality κ in Section 5 by showing that any family of mutually complementary partial orders of a certain kind can be modified in an appropriate way. We need the partial orders to have the property that the set of minimal elements is finite. The maximal chains in these partial orders have size 4. One advantage of this construction is that it is direct rather than inductive. Another advantage is that a similar construction gives the best known lower asymptotic bound in the finite case [10].

Theorem 2. *If κ is an infinite cardinal, then there are κ mutually complementary partial orders on κ. Furthermore, the set of minimal or maximal elements in each of these partial orders is finite and all the maximal chains in each of these partial orders have size* 4.

We need a definition and a lemma before proceeding with the proof.

Definition 1. Let $\{B^m_j : m, j \in \kappa\}$ be a family of subsets of κ such that each $\{B^m_j : m \in \kappa\}$ is a partition. We call this indexed family *saturated* if, for each v, m, there is z such that $z \in B^v_m$ and $v \in B^m_z$.

Lemma 1. *For any infinite cardinal κ, there is a saturated family of subsets of κ.*

Proof. Express κ as the union of an increasing sequence $\{A_n : n \in \omega\}$ where, for al $n \in \omega$, $|A_{n+1} - A_n| = \kappa$. Define partitions $\{\{{}^nB_j^m : m \in \kappa\} : j \in A_n\}$ of A_n such that $(\forall v, m \in A_n)(\exists z \in A_{n+1} - A_n) z \in^{n+1} B_m^v \wedge v \in^{n+1} B_z^m$ using any one-to-one map from A_n^2 to $A_{n+1} - A_n$.

Proof of Theorem 2. Let $X = 2 \times 2 \times \kappa \times \kappa$. We define $\chi : \kappa^2 \to 2$ by letting $\chi(m, x) = 1 \Leftrightarrow 1 + m = x$. Let $\{B_j^m : m, j \in \kappa\}$ be a saturated family of subsets of κ. Let $<_m$ (where $m < \kappa$) be the smallest partial ordering on X which satisfies:

(1) $(j, k) \neq (m, 0) \Rightarrow (1 - \chi(m, k), i, j, k) <_m (0, i, m, 0)$.
(2) $(j, k) \neq (m, 0) \Rightarrow (1, 1 - i, m, 0) <_m (\chi(m, k), i, j, k)$.
(3) $q \in B_j^m, (j, k), (q, p) \neq (m, 0) \Rightarrow (\chi(m, k), i, j, k) <_m (1 - \chi(m, p), r, q, p)$.

We complete the proof with a series of lemmas. The next lemma gives some basic information about each of these partial orders including the fact that they each have four levels.

Lemma 2. *There is a function $lv_m : X \to \{1, 2, 3, 4\}$ which is defined for each $m \in \kappa$ so that*

- $(\forall p_0, p_1 \in X) p_0 <_m p_1 \Rightarrow lv_m(p_0) < lv_m(p_1)$,
- *if $m \neq v$, then $lv_m(p) = 1 \Rightarrow lv_v(p) = 3$ and $lv_m(p) = 4 \Rightarrow lv_v(p) = 2$,*
- *if $lv_m(p) = i$ and $i \leq 3$, then there is $q \in X$ such that $p <_m q$ and $lv_m(q) = i + 1$,*
- *if $lv_m(p) = i$ and $i \geq 2$, then there is $q \in X$ such that $q <_m p$ and $lv_m(q) = i - 1$.*

Proof. In case $(t, u) = (m, 0)$, define $lv_m(i, s, t, u) = 1$ if $i = 1$ and define $lv_m(i, s, t, u) = 4$ if $i = 0$. In case $(t, u) \neq (m, 0)$, define $lv_m(i, s, t, u) = 2$ if $\chi(m, u) = i$ and define $lv_m(i, s, t, u) = 3$ if $\chi(m, u) \neq i$. The definition provides us with an element below or above a given element at an adjacent level since each B_m^v is nonempty and since, for each v, m, there is z such that $z \in B_m^v$ and $v \in B_z^m$. The fact that the order relation is defined on adjacent levels means that the only order relations between elements on nonadjacent levels occur through the transitive closure.

Lemma 3. *If $m \neq v$, then $<_m \cap <_v = \mathrm{id}_X$.*

Proof. Suppose $(i, s, t, u) <_m, <_v (j, p, q, r)$.

Case 1. $\{(t, u), (q, r)\} \cap \{(m, 0), (v, 0)\} = \varnothing$. In both orders $lv(i, s, t, u) = 2$ and $lv(j, p, q, r) = 3$. By condition 3, $q \in B_t^m$ and $q \in B_t^v$ which contradicts the fact that $\{B_t^m : m \in \kappa\}$ is a partition for each $t \in \kappa$.

Case 2. $\{(t, u), (q, r)\} = \{(m, 0), (v, 0)\}$. Suppose that $(t, u) = (m, 0)$ and $(q, r) = (v, 0)$. Now $lv_m(i, s, t, u) = 1$, since it cannot be 4 and so $lv_v(i, s, t, u) = 3$. Thus $lv_v(j, p, q, r) = 4$. Hence $lv_m(j, p, q, r) = 2$. This means that condition 1 applies in $<_v$ and condition 2 applies in $<_m$ to yield $p = s$ and $1 - p = s$. For each v, m, there is z such that $z \in B_m^v$ and $v \in B_z^m$, which is a contradiction.

Case 3. $|\{(t, u), (q, r)\} \cap \{(m, 0)(v, 0)\}| = 1$. Suppose $(t, u) = (m, 0)$. This means that $lv_m(i, s, t, u) = 1$ since it cannot be 4. Thus $lv_v(i, s, t, u) =$

3 and so $lv_v(j, p, q, r) = 4$. This implies that $(q, r) = (v, 0)$ which is a contradiction. Suppose $(q, r) = (m, 0)$. This means that $lv_m(j, p, q, r) = 4$ since it cannot be 1. Thus $lv_v(j, p, q, r) = 2$ and so $lv_v(i, s, t, u) = 1$. This implies that $(t, u) = (v, 0)$, which is a contradiction. The proof is complete.

Lemma 4. *If $m \neq v$, then there is no proper preorder (i.e., no preorder not equal to κ^2) which contains $<_m$ and $<_v$.*

Proof. Any open set in $<_m$ contains a point at level 4. In $<_v$, that point $(0, i, m, 0)$ is at level 2 and so we get $(1 - \chi(v, 1+m), r, q, 1+m)$ in that open set whenever $q \in B^v_m$ by the definition of the partial order between levels 2 and 3. Suppose this open set does not contain all minimal points in $<_v$. Let one of these points be $(1, y, v, 0)$ which is at level 3 in $<_m$. Now choose a $z \in \kappa$ so that $z \in B^v_m$ and $v \in B^m_z$. This is possible by saturation. In particular above, we get both $(1 - \chi(v, 1+m), r, z, 1+m)$ to lie in the open set. Now we know that $(1 - \chi(v, 1+m), r, z, 1+m) <_m (1, y, v, 0)$ since $\chi(m, 0) = 0$ by the definition of the partial order between levels 2 and 3, and so we are done.

Corollary 1. *For any infinite cardinal κ, there are at least κ mutually complementary T_0 topologies on κ.*

3. Lower bounds for equivalence relations

Two equivalence relations $\sim_1$ and $\sim_2$ on a set X are complementary if and only if

- $x \sim_1 y \wedge x \sim_2 y \Rightarrow x = y$,
- all points are equivalent in the transitive closure of $\sim_1 \cup \sim_2$.

Theorem 3. *For any infinite cardinal κ, there is a family of κ many mutually complementary equivalence relations.*

Proof. We say that an ordinal α is congruent to 0 modulo 3 and indicate this fact by $\alpha \equiv 0 \pmod 3$ if $\alpha = \eta + n$ where η is a limit ordinal (possibly the ordinal 0) and $n \in \omega$ and $n \equiv 0 \pmod 3$. Let $\{(a_\xi, b_\xi, c_\xi, d_\xi) \colon 0 < \xi \in \kappa, \xi \equiv 0 \pmod 3\}$ enumerate the 4-tuples of ordinals in κ such that $c_\xi \neq d_\xi$ in such a way that a_ξ, b_ξ, c_ξ and d_ξ are elements of ξ. Now define, by induction on ξ, a family of equivalence relations $\{\sim^\xi_\eta \colon \eta \in \kappa, \xi \leq \kappa, \xi \equiv 0 \pmod 3\}$ so that

(1) $\sim^\xi_\eta$ is an equivalence relation on ξ,

(2) $\alpha \neq \beta$, $x \sim^\xi_\alpha y$, $x \sim^\xi_\beta y \Rightarrow x = y$,

(3) $a_\eta \sim b_\eta$ where $\sim$ is the transitive closure of $\sim^{\eta+3}_{c_\eta} \cup \sim^{\eta+3}_{d_\eta}$,

(4) $\xi < \zeta \Rightarrow \sim^\zeta_\eta \cap \xi^2 = \sim^\xi_\eta$.

We get started by putting all $\sim^3_\eta$ to be the identity relation. Suppose that such equivalence relations $\sim^\xi_\eta$, on $\xi \equiv 0 \pmod 3$ and $\xi > 0$ have been constructed for each $\eta \in \kappa$. We now define the equivalence relations $\sim^{\xi+3}_\eta$ on $\xi + 3$.

If $\eta \notin \{c_\xi, d_\xi\}$, then define $\sim^{\xi+3}_\eta$ to be the smallest equivalence relation which obeys condition (4). That is, we put the next three ordinals into new one element equivalence classes.

Define $\sim^{\xi+3}_{c_\xi}$ to be the smallest equivalence relation which obeys condition (4) such that $a_\xi \sim^{\xi+3}_{c_\xi} \xi$ and $\xi + 1 \sim^{\xi+3}_{c_\xi} \xi + 2$. That is, we put ξ into the same old equivalence class as a_ξ and put $\xi+1$ and $\xi+2$ into a new two element equivalence class.

Define $\sim_{d_\xi}^{\xi+3}$ to be the smallest equivalence relation which obeys condition (4) such that $b_\xi \sim_{d_\xi}^{\xi+3} \xi+2$ and $\xi \sim_{d_\xi}^{\xi+3} \xi+1$. That is, we put $\xi+2$ into the same old equivalence class as b_ξ and put ξ and $\xi+1$ into a new two element equivalence class.

Conditions (1), (2) and (4) are ensured by the construction while the sequence

$$a_\xi \sim_{c_\xi}^{\xi+3} \xi \sim_{d_\xi}^{\xi+3} \xi+1 \sim_{c_\xi}^{\xi+3} \xi+2 \sim_{d_\xi}^{\xi+3} b_\xi$$

implies condition (3). If $\xi \leq \kappa$ is a limit ordinal, then let $\sim_\eta^\xi = \bigcup\{\sim_\eta^\zeta : \zeta < \xi\}$.

For each $\eta \in \kappa$, let $\sim_\eta = \sim_\eta^\kappa$. Condition (2) implies that no two points are equivalent in more than one $\sim_\eta$ while condition (3) implies that, for any two distinct equivalence relations $\sim_{c_\xi}$, $\sim_{d_\xi}$ and, for any two elements $a_\xi, b_\xi \in \kappa$, these two elements are equivalent in the transitive closure of these two equivalence relations. Since we have enumerated all ordered 4-tuples whose last two elements are distinct, we are done.

4. Upper bounds for topologies and T_1 topologies

Theorem 4. *There do not exist κ^+ many mutually complementary topologies on κ.*

Proof. Suppose $\{\tau_\alpha : \alpha \in \kappa^+\}$ were mutually complementary topologies on κ.

- *Case* 1. For κ^+ many α, there is a point which is not closed in τ_α. Suppose that γ_α is another point in the closure of $\{\beta_\alpha\}$. We can assume that β_α and γ_α are fixed and this contradicts the assumption that the join of each two τ_α's is discrete.
- *Case* 2. For at least two α, each point is closed in τ_α. This means that the meet of these two τ_α's contains the cofinite topology which is a contradiction.

Since these two cases exhaust the possibilities, we are done.

Corollary 2. • *The maximum number of mutually complementary topologies on κ is precisely κ.*

- *The maximum number of mutually complementary partial orders on κ is precisely κ.*
- *The maximum number of mutually complementary equivalence relations on κ is precisely κ.*
- *The maximum number of mutually complementary preorders on κ is precisely κ.*

An important cardinal invariant is $\mathbf{p}$ [13] which indicates the least cardinality of a family of subsets of ω with the finite intersection property but for which there is no infinite subset of ω which is almost contained (modulo finite) in each element of the family. Of course $\aleph_1 \leq \mathbf{p} \leq \mathfrak{c}$.

Theorem 5. *If there are more than κ pairwise T_1-complementary topologies on κ, then $\mathbf{p} \leq \kappa < 2^{\mathfrak{c}}$.*

Proof. Suppose that $\{\tau_\alpha : \alpha \in \kappa^+\}$ is a family of pairwise T_1-complementary topologies on κ. Let $\mathscr{U}$ be a free ultrafilter on ω. For each $\beta \in \kappa$, there is at most one $\alpha \in \kappa^+$ such that $\mathscr{U}$ converges to β in τ_α. This means that for all but κ many $\alpha \in \kappa^+$, $\mathscr{U}$ converges nowhere in τ_α.

If $2^c \leq \kappa$, then for all but κ many $\alpha \in \kappa^+$ no ultrafilter on ω converges in τ_α. Let $A \subset \kappa^+$ list these all but κ many elements of κ^+. Let $\alpha_0, \alpha_1 \in A$ be arbitrary. We deduce that ω must be closed in both τ_{α_0} and τ_{α_1} which is impossible. If $\kappa < \mathbf{p}$, choose α_0, α_1 such that $\mathscr{U}$ converges neither in τ_{α_0} nor in τ_{α_1}. Find in τ_{α_0} and find in τ_{α_1} a cover of κ by κ many open sets U such that $U \cap \omega \notin \mathscr{U}$. Let these open sets be listed as $\{U_\alpha^i : \alpha \in \kappa, i \in 2\}$. For each $\alpha \in \kappa$, $i \in 2$, let $Z_\alpha^i = \omega - U_\alpha^i$. Now $\{Z_\alpha^i : \alpha \in \kappa, i \in 2\} \subset \mathscr{U}$. So, we can find $P \in [\omega]^\omega$ which is almost contained in each Z_α^i. Now P is closed in both τ_{α_0} and τ_{α_1}.

Corollary 3. • *The maximum number of mutually T_1-complementary topologies on ω is ω.*

- *It is consistent that the maximum number of T_1-complementary topologies on ω_1 is ω_1 (see p. 127 of [13]).*
- *Under $\mathfrak{p} = \mathfrak{c}$ and $2^\mathfrak{c} = \mathfrak{c}^+$ (e.g. under GCH), if $\kappa \neq \mathfrak{c}$, then the maximum number of mutually T_1-complementary topologies on κ is κ.*

5. Lower bounds for T_1 topologies

Lemma 5. *If $\{\lhd_\alpha : \alpha \in \kappa\}$ are mutually complementary partial orders on an infinite cardinal κ in the lattice of all partial orders and if, for each $\alpha \in \kappa$, the set of $\lhd_\alpha$-minimal elements $F_\alpha \subset \kappa$ is finite, then there are mutually T_1 complementary topologies $\{\sigma_\alpha : \alpha \in \kappa\}$ on $\kappa \times \omega$.*

Proof. Let τ_α be the topology generated by sets of the form $\{\gamma \in \kappa : \delta \lhd_\alpha \gamma\}$ where $\delta \in \kappa$. For each $\alpha, \gamma \in \kappa$, $n \in \omega$ and for each τ_α basic open set U about $\gamma \in \kappa$, let $U^*_{\gamma,n} = (U - \{\gamma\}) \times \omega \cup \{(\gamma, n)\}$. Declare σ_α to be generated by these $U^*_{\gamma,n}$'s and the cofinite topology.

To see joins, let (γ, n), α_0, α_1 be given. There are $U \in \tau_{\alpha_0}$, $V \in \tau_{\alpha_1}$ such that $U \cap V = \{\gamma\}$. Now $U^*_{\gamma,n}$ and $V^*_{\gamma,n}$ are open respectively in σ_{α_0} and σ_{α_1} and have intersection equal to $\{(\gamma, n)\}$.

To see meets, let U be open in both σ_{α_0} and σ_{α_1}. Let $\pi : \kappa \times \omega \to \kappa$ be the projection mapping. Note that, for any $U \in \sigma_\alpha$, $\pi(U) \in \tau_\alpha$. Thus $\pi(U) = \kappa$ so, for each $\alpha \in \kappa$, there is $n \in \omega$ such that $(\alpha, n) \in U$. This means U contains (α, n_α) for each $\alpha \in F_{\alpha_0}$ and some integers n_α. By definition, this means that U contains a cofinite subset of $(\kappa - F_{\alpha_0}) \times \omega$. Similarly U contains a cofinite subset of $(\kappa - F_{\alpha_1}) \times \omega$. Since $F_{\alpha_0} \cap F_{\alpha_1} = \varnothing$, we are done.

Corollary 4. *For each infinite cardinal κ, there are κ many mutually complementary topologies in the lattice of all T_1 topologies on κ.*

Proof. Apply Theorem 2 and Lemma 5.

To proceed further to the construction of more than κ many mutually T_1 complementary topologies on κ, we seem to need a definition which is of independent interest:

Definition 2. $\{X_\gamma : \gamma \in \lambda\} \subset [\lambda]^{<\lambda}$ is said to be a *splitting sequence on λ* if every $X \in [\lambda]^\omega$ is split by each flip beyond some point.

This means

$$(\forall X \in [\lambda]^{\omega})(\exists \alpha \in \lambda)(\forall F \in [\lambda - \alpha]^{<\omega})(\forall \pi\colon F \to 2)|X \cap \bigcap\{X_\gamma^{\pi(\gamma)}\colon \gamma \in F\}| = \omega$$

where we require that each $X_\gamma \subset \gamma$ and indicate X_γ and $\gamma - X_\gamma$ by X_γ^1 and X_γ^0 respectively.

Theorem 6. *If λ is an uncountable regular cardinal, there is an almost disjoint family of κ many subsets of λ (modulo $< \lambda$) and there is a splitting sequence on λ, then there is a family of κ mutually T_1-complementary topologies on λ.*

We need a lemma:

Lemma 6. *If λ is an uncountable regular cardinal and there is a family $\{C_\eta^\alpha\colon \alpha \in \kappa, \eta \in \lambda\}$ of subsets of λ and an almost disjoint family $\{B_\alpha\colon \alpha \in \kappa\}$ of subsets of λ (modulo $< \lambda$) satisfying:*

(1) $(\forall \alpha \in \kappa)(\forall \eta \in \lambda)C_\eta^\alpha$ *is almost equal to* B_α *(modulo* $< \lambda$*),*
(2) $(\forall \gamma \in \lambda)(\forall \alpha_0 \neq \alpha_1 \in \kappa)(\exists \eta \in \lambda)\colon \{\gamma\} = C_\eta^{\alpha_0} \cap C_\eta^{\alpha_1}$,
(3) $(\forall X \in [\lambda]^{\omega})(\forall \alpha \in \kappa)(\exists \delta \in \lambda)(\forall \Lambda \in [\lambda - \delta]^{<\omega})|\bigcap\{C_\eta^\alpha\colon \eta \in \Lambda\} \cap X| = \aleph_0$,

then there is a family of κ mutually T_1-complementary topologies on λ.

Proof of Lemma 6. For each $\alpha \in \kappa$, let the topology τ_α have the subbase $\{C_\eta^\alpha - F\colon \eta \in \lambda, F \in [\lambda]^{<\omega}\}$. Condition (2) says that $\{\gamma\}$ is open in $\tau_{\alpha_0} \vee \tau_{\alpha_1}$ and thus that the join is discrete.

If the meet is not cofinite, suppose that $\alpha_0, \alpha_1 \in \kappa$ and that V is a coinfinite set which is open in both τ_{α_0} and τ_{α_1}. Let $X \in [\lambda]^{\omega}$ be disjoint from V. Now V contains some $\bigcap\{C_\eta^{\alpha_0}\colon \eta \in \Gamma\} - G$ where $\Gamma \in [\lambda]^{<\omega}$ and $G \in [\lambda]^{<\omega}$. Choose $\delta \in \lambda$ from condition (3) for $\alpha_1 \in \kappa$. Now choose $\beta \in \bigcap\{C_\eta^{\alpha_0}\colon \eta \in \Gamma\} - \bigcup\{C_\eta^{\alpha_1}\colon \eta < \delta\} - G$ which can be done since each $C_\eta^{\alpha_i}$ is almost equal to B_{α_i}. Since $\beta \in V$ and V is open in τ_{α_1}, there is $\Lambda \in [\lambda]^{<\omega}$ and $F \in [\lambda]^{<\omega}$ so that $\beta \in \bigcap\{C_\eta^{\alpha_1}\colon \eta \in \Lambda\} - F \subset V$. Since $\Lambda \cap \delta = \varnothing$, condition (3) implies that $Z = \bigcap\{C_\eta^{\alpha_1}\colon \eta \in \Lambda\} \cap X$ is an infinite set. Now $Z - F \subset V$ and $Z \subset X$ contradicts the fact that $V \cap X = \varnothing$.

Proof of Theorem 6. Let $\{B_\alpha\colon \alpha \in \kappa\}$ be an almost disjoint family of subsets of λ (modulo $< \lambda$). Let $\{X_\gamma\colon \gamma \in \lambda\}$ be a splitting sequence. Since $\kappa \leq 2^\lambda$, let $\chi\colon \kappa \times \lambda \to 2$ be such that $(\forall \alpha_0, \alpha_1 \in \kappa)(\exists \eta \in \lambda)\colon \chi(\alpha_0, \eta) \neq \chi(\alpha_1, \eta)$. Let $\psi_0, \psi_1\colon \lambda \to \lambda$ be so that for each $\gamma_0, \gamma_1 \in \lambda$ there are cofinally many $\eta \in \lambda$ such that $\psi_0(\eta) = \gamma_0$ and $\psi_1(\eta) = \gamma_1$. Define

$$C_\eta^\alpha = \{\psi_0(\eta)\} \cup (B_\alpha - \eta) \cup X_\eta^{\chi(\alpha, \psi_1(\eta))}$$

for each $\eta \in \lambda$ and $\alpha \in \kappa$. We apply Lemma 6.

To show condition (2), suppose that $\gamma \in \lambda$ and $\alpha_0 \neq \alpha_1 \in \kappa$ are given. We choose $\eta_0 \in \lambda$ such that $B_{\alpha_0} \cap B_{\alpha_1} \subset \eta_0$, choose γ_1 so that $\chi(\alpha_0, \gamma_1) \neq \chi(\alpha_1, \gamma_1)$ and choose $\eta \geq \eta_0$ so that $\psi_0(\eta) = \gamma$ and $\psi_1(\eta) = \gamma_1$. Since $X_\eta^0 \cup X_\eta^1 \subset \eta$, we have that $\{\gamma\} = C_\eta^{\alpha_0} \cap C_\eta^{\alpha_1}$.

To show condition (3), suppose that $X \in [\lambda]^{\omega}$ and $\alpha \in \kappa$. Since $\{X_\gamma\colon \gamma \in \lambda\}$ is a splitting sequence, there is a $\delta \in \lambda$ so that every flip from $\{X_\gamma\colon \gamma \in \lambda - \delta\}$ intersects X in an infinite set. Now if $\Lambda \in [\lambda - \delta]^{<\omega}$, then $\bigcap\{C_\eta^\alpha\colon \eta \in \Lambda\} \subset \bigcap\{X_\eta^{\chi(\alpha, \psi_1(\eta))}\colon \eta \in \Lambda\}$ and even the latter intersects X in an infinite set. The theorem is proved.

Corollary 5. *Under CH, there are $2^{\aleph_1}$ many mutually T_1-complementary topologies on ω_1.*

Proof. The complete binary tree of height ω_1 has c many vertices and $2^{\aleph_1}$ many branches. A splitting sequence for ω_1 can be readily constructed under CH by transfinite induction. In fact, suppose that $\{\mathcal{M}_\alpha : \alpha \in \omega_1\}$ is a sequence of elementary submodels of $H(\mathfrak{c}^+)$ such that

- $\alpha \in \beta \Rightarrow \mathcal{M}_\alpha \in \mathcal{M}_\beta$,
- $\bigcup\{\mathcal{M}_\alpha : \alpha \in \omega_1\} \supset [\omega]^\omega$,
- $(\forall \alpha \in \omega_1)(\exists r \in \mathcal{M}_{\alpha+1} \cap 2^\omega) r$ splits every real in $\mathcal{M}_\alpha$ (so any Cohen real over $\mathcal{M}_\alpha$ works).

Thinking of the reals as the power set of α, there is some $X_\alpha \in \mathcal{M}_{\alpha+1}$ which splits every infinite subset of α which belongs to $\mathcal{M}_\alpha$. Thus $\{X_\alpha : \alpha \in \omega_1\}$ is the desired splitting family.

Corollary 6. *It is consistent with any cardinal arithmetic that there are $2^{\aleph_1}$ many mutually T_1-complementary topologies on ω_1.*

Proof. The proof of Corollary 4 shows that there is a splitting sequence on ω_1 in any model which is obtained by adding $\aleph_1$ Cohen reals to a model of set theory.

Problem 1. Can one establish, in ZFC, that there are $\mathfrak{c}^+$ many (maybe even $2^{\mathfrak{c}}$ many) mutually T_1-complementary topologies on $\mathfrak{c}$?

Problem 2. Are there infinitely many mutually T_1-complementary (completely regular) Hausdorff spaces?

6. Maximal mutually complementary families

Theorem 7. *If $3 \leq \nu \leq \kappa$, then there is a maximal family of ν many mutually complementary equivalence relations on κ. In fact, these families are maximal even among preorders.*

Proof. Let λ be a cardinal which equals ν if ν is infinite and otherwise equals $\nu - 1$. We say that two functions $f_0, f_1 \in \kappa^\lambda$ are *orthogonal* if $|f_0 \cap f_1| \leq 1$. By induction on $\alpha \in \kappa$, we can find $\lambda + 1$ many partitions $\{\{f_\alpha^\gamma : \alpha \in \kappa\} : -1 \leq \gamma < \lambda\}$ of $\lambda \times \kappa$ into functions in κ^λ in such a way that, for each $-1 \leq \gamma < \lambda$, $\{f_\alpha^\gamma : \alpha \in \kappa\}$ is an orthogonal family.

To see this, note that if $\lambda \neq \kappa$, then $\lambda \times \kappa = \bigcup\{\lambda \times X_\xi : \xi \in \kappa\}$ where $\{X_\xi : \xi \in \kappa\}$ is a partition of κ into sets of size λ and so each $\lambda \times X_\xi$ can be partitioned into functions forming an orthogonal family separately. So assume $\lambda = \kappa$ and proceed by induction on $\beta \in \kappa$ and suppose that partitions of $\kappa \times \kappa$ into functions $\{\{f_\alpha^\gamma : \alpha \in \kappa\} : -1 \leq \gamma < \beta\}$ have been constructed and that the family $\{\{f_\alpha^\gamma : \alpha \in \kappa, -1 \leq \gamma < \beta\}$ is an orthogonal family. To construct $\{f_\alpha^\beta : \alpha \in \kappa\}$, let $\{(\mu_\xi^0, \mu_\xi^1) : \xi \in \kappa\}$ enumerate $\kappa \times \kappa$ and proceed by induction. If $\{f_\alpha^\beta : \alpha \in \zeta\}$ have been constructed, let $f_\zeta^\beta(\mu_\zeta^0) = \mu_\zeta^1$. Extend this definition of f_ζ^β so that it is a total function which is disjoint from each element of $\{f_\alpha^\beta : \alpha \in \zeta\}$ and is orthogonal to each f_α^γ for $\alpha \in \kappa$, $\gamma \in \beta$. This can be done since, if f_ζ^β has been defined on fewer than κ many elements, then the inductive hypothesis of orthogonality guarantees that f_ζ^β so far intersects fewer than κ many elements of $\{f_\alpha^\gamma : \alpha \in \kappa, -1 \leq \gamma < \beta\}$. Since the family $\{f_\alpha^\beta : \alpha \in \zeta\}$ also has size less than κ, it is possible to extend f_ζ^β on any one element and preserve all the inductive requirements.

For each $0 \leq \gamma < \lambda$, we construct equivalence relations $\{\sim_\gamma : \gamma \in \lambda\}$ on $(\lambda \times \kappa) \cup \{\infty\}$ by declaring each $f_\alpha^\gamma - \{(\gamma, f_\alpha^\gamma(\gamma))\}$ to be an equivalence class in $\sim_\gamma$ and declaring $(\{\gamma\} \times \kappa) \cup \{\infty\}$ to be an equivalence class in $\sim_\gamma$.

We construct $\sim_{-1}$ by declaring each f_α^{-1} to be an equivalence class in $\sim_{-1}$ as well as declaring $\{\infty\}$ to be an equivalence class in $\sim_{-1}$.

To see that the meets are all the identity equivalence relation, note that restricted to $\lambda \times \kappa$ orthogonality applies. Suppose that A is closed under $\sim_\gamma \vee \sim_\delta$ where $\gamma \neq \delta \in \lambda$ and $\gamma \neq -1$. Since A is closed under $\sim_\delta$, either $\infty \in A$ or $A \cap (\{\gamma\} \times \kappa) \neq \varnothing$. Since A is closed under $\sim_\gamma$ and $\gamma \neq -1$, in either case, $\infty \in A$. Since this reasoning applies equally to A^c, we get a contradiction.

To see that this family is maximal, suppose that $\lhd$ is another preorder which is complementary to each $\sim_\gamma$, where $-1 \leq \gamma < \lambda$. Since ∞ is an equivalence class in $\sim_{-1}$, there must be some $(\gamma, \alpha) \in \lambda \times \kappa$ such that either $(\gamma, \alpha) \lhd \infty$ or $\infty \lhd (\gamma, \alpha)$. However $(\gamma, \alpha) \sim_\gamma \infty$ and either way that is a contradiction.

This construction works for any $\lambda \geq 2$. Actually it works for $\lambda = 1$ as well, but we obtain only $\sim_{-1}$ as the identity relation and $\sim_0$ with one equivalence class.

D. Dikranjan and A. Policriti [12] have recently shown that there are many families of two mutually complementary equivalence relations on a finite set.

Problem 3. What are the possible cardinalities of maximal families of mutually complementary families of partial orders (or T_0 topologies)?

We now of no finite maximal mutually complementary families of partial orders of cardinality greater than two on an infinite set. We know of no countably infinite maximal mutually complementary families of partial orders on an uncountable set. There is an easy example of a maximal family of two mutually complementary equivalence relations on any uncountable set. Let $\sim_0$ on $\kappa \times 2$ have two equivalence classes $\kappa \times \{0\}$ and $\kappa \times \{1\}$. Let $\sim_1$ on $\kappa \times 2$ have each $\{\alpha\} \times 2$ as an equivalence class.

Problem 4. What are the possible cardinalities of maximal families of mutually complementary families of T_1 topologies?

Here we know of no examples of maximal mutually complementary families except by appealing to Zorn's lemma and using the upper bounds established in Section 4.

Definition 3. Suppose $<$ is a preorder on κ. Let $A \subset \kappa$. Define $\uparrow_< (A) = \{b \in \kappa : (\exists a \in A) b > a\}$. If we have $(\forall A \neq \varnothing) \uparrow_< (\uparrow_{<'} (\uparrow_< (\uparrow_{<'} (A)))) = \kappa$, then we say that $<$ and $<'$, in that order, are 4-complementary. If we have $(\forall A \neq \varnothing) \uparrow_< (\uparrow_{<'} (\uparrow_< (A))) = \kappa$, then we say that $<$ and $<'$, in that order, are 3-complementary. If we have $(\forall A \neq \varnothing) \uparrow_< (\uparrow_{<'} (A)) = \kappa$, then we say that $<$ and $<'$, in that order, are 2-complementary.

Note that the preorders constructed in Theorems 1, 2 and 3 are mutually 4-complementary.

Problem 5. What are the possible sizes of mutually 3-complementary (mutually 2-complementary) preorders (partial orders) (equivalence relations)?

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Department of Mathematics, York University, North York, Ontario, Canada M3J 1P3

E-mail address: `steprans@mathstat.yo`

E-mail address: `watson@vml.yorku.ca`