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COMPLETION THEOREM FOR COHOMOLOGICAL DIMENSIONS

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ABSTRACT. We prove that for every separable metrizable space X with $\dim_G X \leq n$, there exists a metrizable completion Y of X with $\dim_G Y \leq n$ provided that G is either a countable group or a torsion group, and with $\dim_G Y \leq n+1$ if G is an arbitrary group.

1. INTRODUCTION

This paper was inspired by the general plan to construct cohomological dimension theory parallel to the theory of covering dimension. We restrict our considerations to metrizable spaces.

We call a space Y an absolute extensor of a space X if every mapping $f\colon A \to Y$, where A is a closed subset of X, extends over X; we write then $Y \in AE(X)$. We define the cohomological dimension $\dim_G X$ of a space X with a coefficient group G as

$$\min\{n\colon\ K(G, n) \in AE(X)\},$$

where $K(G, n)$ stands for the Eilenberg-MacLane CW complex (see, for example, [5]). Note that Y. Kodama proved in [3] that every mapping $f\colon X \to K$ from a closed subset A of a metrizable space X to a CW complex K extends over an open set $U \subseteq X$ containing A. For a deeper discussion of cohomological dimensions in the realm of metrizable spaces we refer the reader to [1] and [2].

It is known as the completion theorem for covering dimension that for every metrizable space X with $\dim X = n$, there exists a completely metrizable space Y containing X with $\dim Y = n$. In [4] L. R. Rubin and P. J. Schapiro proved that the analogous theorem holds for cohomological dimension with the coefficient group $G = \mathbb{Z}$, where $\mathbb{Z}$ denotes integers, provided X is metrizable separable; they derived this theorem from their generalization of the Edwards-Walsh theorem.

We give the direct proof of that completion theorem and its generalization to every group G which is either countable or torsion. We also prove that for any group G and separable metrizable space X with $\dim_G X = n$, there exists a completely metrizable space Y containing X with $\dim_G Y = n+1$. The

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results show that Conjecture 3 of [2] is true, and answer partially Problems 3 and 4.

2. Proofs

We shall start with the main theorem.

Theorem 1. *Let K be a countable CW complex. For every metrizable separable space X such that $K \in AE(X)$, there exists a metrizable completion Y of X such that $K \in AE(Y)$.*

The following simple lemma established by Walsh will play a key role in the proof.

Lemma 1 (Walsh [5]). *Every mapping $f\colon A \to K$ from a subspace A of a metrizable separable space Z to a CW complex K is homotopic to a mapping $f^*\colon A \to K$ which extends over an open subspace $U \subseteq Z$ containing A.*

Lemma 2. *Let K be a CW complex, Z a metrizable separable space, and X its subspace such that $K \in AE(X)$. Every mapping $g\colon C \to K$ from a closed subset $C \subseteq Z$ extends over an open set $U \subseteq Z$ such that $C \cup X \subseteq U$.*

Proof. Extend g to a g^* over the closure of an open V containing C, and next, $g^*|\mathrm{bd}\, V \cap X$ to an h over X. It is easily seen that $f(x) = g^*(x)$ for $x \in V$ and $f(x) = h(x)$ for $x \in X - V$ is a well-defined mapping of $V \cup X$. Therefore by Lemma 1, there exists a mapping homotopic to f which is extendable over an open set $U \subseteq Z$ such that $C \cup X \subseteq V \cup X \subseteq U$, and so is g.

Proof of Theorem 1. Assume that $X \subseteq Q$, where Q stands for the Hilbert cube; we shall construct a G_δ-subspace $Y \subseteq Q$ containing X such that $K \in AE(Y)$. To this end, we shall first define inductively countable families $\mathfrak{U}_n$ of open subspaces of Q containing X and families of mappings $\{f_U\colon U \in \mathfrak{U}_n\}$, where $f_U\colon U \to K$.

Let $\{D_k\colon k \in \mathbb{N}\}$ be a family of closed subsets of Q with the property that for any closed $C \subseteq Q$ and open $V \subseteq Q$ containing C, there is a $k \in \mathbb{N}$ such that $C \subseteq \mathrm{int}\, D_k \subseteq D_k \subseteq V$. The set $[D_k, K]$ of all homotopy classes of mappings from D_k to K is countable; that is, $[D_k, K] = \{[g^{(0)}_{k,i}]\colon i \in \mathbb{N}\}$. By Lemma 2, every $g^{(0)}_{k,i}$ extends over an open subspace $U^{(0)}_{k,i} \subseteq Q$ containing $D_k \cup X$; denote the extension by $f^{(0)}_{k,i}$. Set

$$\mathfrak{U}_0 = \{U^{(0)}_{k,i}\colon k, i \in \mathbb{N}\},$$

and $f_U = f^{(0)}_{k,i}$ for $U = U^{(0)}_{k,i}$. By the Homotopy Extension Theorem, we have

(1) every mapping $f\colon D_k \to K$ extends over U to a mapping homotopic to f_U for some $U \in \mathfrak{U}_0$.

Suppose $\mathfrak{U}_n$ and mappings f_U, $U \in \mathfrak{U}_n$, are constructed. For every $U \in \mathfrak{U}_n$, D_l, D_m, D_p, and D_q such that $D_l \subseteq \mathrm{int}\, D_m$, $D_p \subseteq \mathrm{int}\, D_q$, and $D_q \cap D_l \subseteq U$, consider the set

$$[(D_l \cap U) \cup (U - \mathrm{int}\, D_q)] \cup (D_p \cap D_m);$$

arrange these sets into a sequence $D^{(n+1)}_k$, $k \in \mathbb{N}$. Observe that for every $D^{(n+1)}_k = [(D_l \cap U) \cup (U - \mathrm{int}\, D_q)] \cup (D_p \cap D_m)$, we have $[(D_l \cap U) \cup (U - \mathrm{int}\, D_q)] \cap$

$(D_p \cap D_m) = D_p \cap D_l$, and $D_k^{(n+1)}$ is a closed subset of $Z_k = U \cup \operatorname{int} D_q$; recall that $D_q \cap D_l \subseteq U$. There is only a countable number of mappings $h_{k,i}$: $D_p \cap D_m \to K$, $i \in \mathbb{N}$, such that $h_{k,i}|D_p \cap D_l = f_U|D_p \cap D_l$ up to homotopy relative $D_p \cap D_l$. Let $g_{k,i}^{(n+1)}(x) = h_{k,i}(x)$ for $x \in D_p \cap D_m$ and $g_{k,i}^{(n+1)}(x) = f_U(x)$ for $x \in [(D_l \cap U) \cup (U - \operatorname{int} D_q)]$. By Lemma 2, $g_{k,i}^{(n+1)}$ extends over an open set $U_{k,i}^{(n+1)} \subseteq Z_k = U \cup \operatorname{int} D_q$ containing X; denote the extension by $f_{k,i}^{(n+1)}$.

Set $\mathfrak{U}_{n+1} = \{U_{k,i}^{(n+1)}: k,\ i \in \mathbb{N}\}$, and $f_U = f_{k,i}^{(n+1)}$ for $U = U_{k,i}^{(n+1)}$. By the Homotopy Extension Theorem, we have

(2) every mapping h: $D_k^{(n+1)} \to K$ such that h is homotopic to f_U in $[(D_l \cap U) \cup (U - \operatorname{int} D_q)]$ extends over V to a mapping homotopic to f_V for some $V \in \mathfrak{U}_{n+1}$.

Let $Y = \bigcap_{n=0}^{\infty} \bigcap \mathfrak{U}_n$. We have to show that $K \in AE(Y)$, i.e., that every f: $A \to K$, where A is a closed subset of Y, is extendable over Y. By virtue of Lemma 1 and the Homotopy Extension Theorem, we can assume that f extends over an open set $U \subseteq Q$; denote the extension by the same letter. Consider a sequence of sets $D_{l(0)}, D_{l(1)}, \ldots$ such that $D_{l(n)} \subseteq \operatorname{int} D_{l(n+1)}$ for $n \in \mathbb{N}$ and $U = \bigcup\{D_{l(n)}: n \in \mathbb{N}\}$, and a sequence of sets $D_{p(0)}, D_{p(1)}, \ldots$ such that $D_{p(n+1)} \subseteq \operatorname{int} D_{p(n)}$ for $n \in \mathbb{N}$ and $\operatorname{cl}_Q A = \bigcap\{D_{p(n)}: n \in \mathbb{N}\}$.

We shall define inductively sets $U_n \in \mathfrak{U}_n$ and mappings f_n: $U_n \to K$ such that

$$D_{p(n)} \cap D_{l(n)} \subseteq U_n, \tag{3}$$

$$f_n \text{ is homotopic to } f_{U_n}, \tag{4}$$

$$f_n(x) = f_{n+1}(x) \quad \text{for every } x \in Y \cap [(Q - \operatorname{int} D_{p(n)}) \cup D_{l(n)}], \tag{5}$$

$$f_n(x) = f(x) \quad \text{for } x \in D_{p(n)} \cap D_{l(n)}. \tag{6}$$

Let $U_0 \in \mathfrak{U}_0$ be such that $f|D_{l(0)}$ extends over U_0 to a mapping f_0 homotopic to f_{U_0} (see (1)). Suppose U_n and f_n are defined. Let $D_l = D_{l(n)}$, $D_m = D_{l(n+1)}$, $D_p = D_{p(n+1)}$, $D_q = D_{p(n)}$. Consider $D_k^{(n+1)} = [(D_l \cap U_n) \cup (U_n - \operatorname{int} D_q)] \cup D_p \cap D_m$ ($D_q \cap D_l \subseteq U_n$ by condition (3) of the inductive assumption), and the mapping h: $D_k^{(n+1)} \to K$ defined by letting $h(x) = f_n(x)$ for $x \in [(D_l \cap U_n) \cup (U_n - \operatorname{int} D_q)]$ and $h(x) = f(x)$ for $x \in D_p \cup D_m$. By virtue of (3) and (6) of the inductive assumption h is well defined; by (2), h extends over a $U_{n+1} \in \mathfrak{U}_{n+1}$ to a mapping f_{n+1} homotopic to $f_{U_{n+1}}$. It is easily seen that (3)–(6) are satisfied.

By (5), the sequence $(f_n)_{n=0}^{\infty}$ determines a mapping f^*: $Y \to K$ which is an extension of f by (6).

Corollary 1. *Let G be a countable abelian group. Every separable metrizable space X has a completion Y with $\dim_G Y = \dim_G X$.*

In [1] J. Dydak established the following theorem.

Theorem. *Let G be an abelian group. Then*

$$\dim_G X = \max\{\dim_{\operatorname{Tor} G} X,\ \dim_{G/\operatorname{Tor} G} X\}, \tag{7}$$

(8) $$\dim_G X = \max\{\dim_H X\colon\ H \in \sigma(G)\}\,,$$

where $\sigma(G)$ *is a countable family of countable groups, provided* G *is a torsion group*

(9) *there exists a countable group* $H(G)$ *such that* $\dim_G X \leq \dim_{H(G)} X \leq \dim_G X + 1$ *provided* G *is torsion free*

for every metrizable space X.

By virtue of this theorem, Corollary 1 and the monotonicity of cohomological dimension (see [5]), we obtain the following corollary.

Corollary 2. *Let* X *be a separable metrizable space, and* G *an abelian group. Then there exists a completion* Y *of* X *with* $\dim_G Y \leq \dim_G X + 1$, *and with* $\dim_G Y = \dim_G X$ *provided* G *is a torsion group.*

We do not know whether for any abelian group G and separable metrizable space X there exists a countable abelian group H such that $\dim_H X \leq \dim_G X$ and $\dim_G Y \leq \dim_H Y$ for every separable metrizable space Y. Of course, the positive answer to this question would give the completion theorem for an arbitrary abelian group G.

Added in Proof

The author is indebted to L. R. Rubin for pointing out a mistake in the introduction; namely, his paper with P. J. Schapiro ([4]) contains the completion theorem for integer cohomological dimension under the assumption that X is a metrizable, not necessarily separable, space.

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