NON-SINGULAR MODULES AND R -HOMOGENEOUS MAPS

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ABSTRACT. A non-singular R -module M is a ray for the class of all non-singular modules if every R -homogeneous map from M into a non-singular module is additive. Every essential extension of a non-singular locally cyclic module is a ray. We investigate the structure of rays, and determine those semi-prime Goldie-rings for which all non-singular modules are rays and those rings for which the only rays are essential extensions of locally cyclic modules.

1. INTRODUCTION

In a recent paper, one of the authors studied endomorphical modules over a principal ideal domain R . These are precisely those modules M for which every R -homogeneous map $M \rightarrow M$ is an endomorphism of M . The discussion of endomorphical modules has its origins in the theory of near rings since the collection of R -homogeneous maps $M \rightarrow M$ forms a near ring. Fuchs, Maxson, and Pilz investigated the class of rings for which all R -modules are endomorphical in [1]. While they were not able to describe this class of rings completely, they showed in [1, Theorem II.8] that it contains all rings of the form $\prod_{i \in I} \text{Mat}_{n_i}(R)$ such that $n_i \geq 2$ for all $i \in I$. If we try to find rings for which the class of endomorphical modules is as small as possible, we observe that the class of endomorphical modules always contains the locally cyclic modules [3, Corollary 3.2]. Here an R -module M is *locally cyclic* if, for $x, y \in M$, there are $a \in M$ and $r, s \in R$ with $x = ar$ and $y = as$. There are rings for which all endomorphical modules are locally cyclic; for instance, an abelian group G is endomorphical if and only if it is locally cyclic [3, Propositions 2.4 and 2.7 and Theorem 3.4].

It is the goal of this paper to investigate further the relationship between locally cyclic modules and endomorphical modules, especially in the case that R is not necessarily a principal ideal domain. The methods used in [3] for the discussion of endomorphical abelian groups do not carry over to the generality of the rings under consideration in this paper. We show in Section 2 that there exists a ring R without zero-divisors that has a right ideal I which is not endomorphical. On the other hand, every ideal in a commutative integral domain is endomorphical, but not necessarily locally cyclic.

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We focus our attention mostly on non-singular modules. In Theorem 2.1, we show that a right non-singular ring R is a semi-prime right Goldie-ring if and only if the following conditions are equivalent for every non-singular right R -module M :

- (a) M contains an essential locally cyclic submodule.
- (b) M contains an essential cyclic submodule.
- (c) M is isomorphic to a submodule of its maximal right ring of quotients.

We consider R -modules M and N , and call M an N -ray if every R -homogeneous map $\alpha: M \rightarrow N$ is an element of $\text{Hom}_R(M, N)$. Given a class $\mathcal{C}$ of R -modules, we say that M is a ray for $\mathcal{C}$ if it is an N -ray for all $N \in \mathcal{C}$. Proposition 3.1 establishes that a non-singular module M which has a non-zero essential locally cyclic submodule is a ray for any class $\mathcal{C}$ of non-singular R -modules. We show that there is a semi-prime right Goldie-ring R such that every non-singular right R -module is a ray for the class of all non-singular modules (Theorem 3.5 and Corollary 3.6), but has the additional property that not all modules are endomorphal. Moreover, we describe the semi-prime right Goldie-rings R for which non-singular modules, which are rays for the class of non-singular modules, are submodules of $Q(R)$ (Theorem 3.4). Theorems 3.4 and 3.6, in particular, show that the class of endomorphal modules may be very large even if not all R -modules are endomorphal.

2. NON-SINGULAR LOCALLY CYCLIC MODULES AND GOLDIE-RINGS

Let R be a ring. The word *module* denotes a unital right R -module. The module M is *non-singular* if $Z(M) = \{x \in M \mid xI = 0 \text{ for some essential right ideal } I \text{ of } R\} = 0$. The ring R itself is (*right*) *non-singular* if it is non-singular as a right R -module. In particular, $Z(R)$ is a two-sided ideal of R . We define the *second singular submodule* $Z_2(M)$ by $Z_2(M)/Z(M) = Z(M/Z(M))$. It is well known that $M/Z_2(M)$ is a non-singular module. Furthermore, we observe $MZ_2(R) = 0$ for all non-singular R -modules M . Therefore, every non-singular R -module can be viewed as an $R/Z_2(R)$ -module, and vice-versa. Therefore, we restrict our discussion in this paper to the case that R is a right non-singular ring. The module M has *Goldie-dimension* 1 if it is non-zero and every non-zero submodule of M is essential in M . In the following, the symbol $E(M)$ denotes the injective hull of M .

We want to remind the reader that a ring R is a *semi-prime right Goldie-ring* if it does not contain any non-zero nilpotent ideals and every direct sum of non-zero right ideals has only finitely many summands. In particular, $Z(R) = 0$ for a semi-prime right Goldie-ring. These rings are precisely those which have a semi-simple Artinian maximal right quotient ring, $Q(R)$. In particular, $Q(R)$ is the classical right ring of quotients of R [2]. For further details on Goldie-rings, see [5].

Theorem 2.1. *The following conditions are equivalent for a right non-singular ring R :*

- (a) R is a semi-prime right Goldie-ring.
- (b) *The following conditions are equivalent for an R -module M :*
 - (i) M contains an essential non-singular cyclic submodule.
 - (ii) M contains an essential non-singular locally cyclic submodule.
 - (iii) M is isomorphic to a submodule of $Q(R)$.

Proof. (a) $\Rightarrow$ (b): The implication (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii): We show that the injective hull of M is isomorphic to a submodule of $Q(R)$. For this, it is enough to show that M contains an essential submodule which is isomorphic to a right ideal I of R . Then, $E(M) = E(I) \subseteq E(R) = Q(R)$. Suppose that U is an essential locally cyclic submodule of M . Since $E(M) = E(U)$, no generality is lost if we assume that M itself is locally cyclic.

Assume that M has infinite Goldie-dimension. If n is the right Goldie-dimension of R , then M contains $n+1$ non-zero cyclic submodules $a_0R, \dots, a_nR$ whose sum is direct. Since M is locally cyclic, there are $a \in M$ and $r_0, \dots, r_n \in R$ with $a_i = ar_i$ for $i = 0, \dots, n$. Since $\bigoplus_{i=0}^n a_iR \subseteq aR$, we obtain that aR has Goldie-dimension at least $n+1$. On the other hand, if I is the annihilator of a , then I is not essential in R since M is non-singular. There is a non-zero right ideal J of R such that $I \oplus J$ is essential in R . Since R/I is non-singular and $[R/I]/[(I \oplus J)/I] \cong R/(I \oplus J)$ is singular, we obtain that $(J \oplus I)/I$ is an essential submodule of R/I . Thus, aR contains an essential submodule isomorphic to J , and $n \geq \dim J = \dim aR \geq n+1$, which is not possible. Therefore, the Goldie-dimension of M has to be finite.

In particular, we can choose non-zero cyclic submodules $b_1R, \dots, b_sR \subseteq M$ of M whose sum is direct and essential in M . Using the arguments of the previous paragraph, there is $b \in M$ such that $\bigoplus_{i=1}^s b_iR \subseteq bR$. Thus, bR is an essential submodule of M . By what has been shown, bR contains an essential submodule which is isomorphic to a right ideal of R .

(iii) $\Rightarrow$ (i): Assume that M is a submodule of $Q(R)$. Choose a submodule V of $Q(R)$ such that $M \cap V = 0$ and $M \oplus V$ is essential in $Q(R)$. Suppose we have shown that $M \oplus V$ contains an essential cyclic submodule $U = xR$. Write $x = a + b$ with $a \in M$ and $b \in V$. Then, $xR \subseteq aR \oplus bR$ is an essential submodule of $M \oplus V$. Therefore, $(M \oplus V)/(aR \oplus bR) \cong (M/aR) \oplus (V/bR)$ is singular. Since M is non-singular, aR is essential in M . We may, hence, assume that M is essential in $Q(R)$.

Observe that R is essential in $Q(R)$. Therefore, $M \cap R$ is an essential right ideal of R . Thus, $M \cap R$ contains dR for some regular element d of R . Since dR is essential in R , we obtain that dR is an essential submodule of $Q(R)$ inside M , i.e., dR is essential in M . Thus, M satisfies (i).

(b) $\Rightarrow$ (a): Let I be an essential right ideal of R . Since I is an R -submodule of $Q(R)$, it contains an essential cyclic submodule dR for some $d \in I$ by the equivalence of (i) and (iii) in (b). Therefore, the family of essential right ideals of R contains a cofinal subfamily of cyclic ideals. Since R is a non-singular ring, it is a semi-prime right Goldie-ring by [2].

However, $Q(R)$ need not be locally cyclic even if R is a semi-prime right Goldie-ring.

Theorem 2.2. *The following conditions are equivalent for a ring R such that R is a semi-prime right Goldie-ring:*

(a) $Q(R)$ is locally cyclic.

(b) R is a left Goldie-ring.

Proof. (b) $\Rightarrow$ (a): Suppose that x and y are elements of $Q(R)$. There are $r, s \in R$ and a regular element d of R with $x = d^{-1}r$ and $y = d^{-1}s$ since

R satisfies the left Ore-condition as a semi-prime left Goldie-ring. This proves that $Q(R)$ is a locally cyclic R -module.

(a) $\Rightarrow$ (b): It suffices to show that R satisfies the left Ore-condition with respect to its regular elements. Since left and right quotient rings are isomorphic if they exist, R has to be a left Goldie-ring. Let $r, c \in R$ with c a regular element of R . Because $Q(R)$ is a locally cyclic right R -module, there are $a \in Q(R)$ and $s, t \in R$ with $rc^{-1} = as$ and $1 = at$. Write $a = uv^{-1}$ for $u, v \in R$ with v regular in R .

Since $1 = uv^{-1}t$, the element t is a right regular element of R . The fact that R is a right Goldie-ring yields that t is regular in R . Therefore, $u = t^{-1}v$ is a regular element of R , and $rc^{-1} = uv^{-1}s = t^{-1}vv^{-1}s = t^{-1}s$. This yields $tr = sc$ for some $s, t \in R$ with t regular; and R satisfies the left Ore-condition.

3. NON-SINGULAR MODULES AND RAYS

In this section, we investigate how locally cyclic non-singular modules and rays are related for the class of rights discussed in Section 2.

Proposition 3.1. *Let R be a ring.*

- (a) *A locally cyclic R -module is a ray for any class $\mathcal{C}$ of R -modules.*
- (b) *If an R -module M is a ray for a class $\mathcal{C}$ of non-singular R -modules, then so is every essential extension of M .*

Proof. Since (a) is an immediate consequence of the definitions, it remains to show (b). Suppose N is a module which contains M as an essential submodule of $E(M)$. Consider $x, y \in N$. There is an essential right ideal I of R such that $xI, yI \subseteq M$. If $\phi: M \rightarrow X$ for some $X \in \mathcal{C}$ is R -homogeneous, then $\phi(x+y)i = \phi(xi+yi) = \phi(xi) + \phi(yi)$ for all $i \in I$. Therefore,

$$(\phi(x+y) - \phi(x) - \phi(y))I = 0.$$

Since X is non-singular, we have $\phi(x+y) = \phi(x) + \phi(y)$.

Corollary 3.2. *Let R be a semi-prime right Goldie-ring. Submodules of $Q(R)$ are rays for the class of non-singular modules.*

Proof. Suppose that there are $r, d \in R$ such that d is regular in R and $rd^{-1} \in Z(Q(R))$. Then, $r \in R \cap Z(Q(R)) \subseteq Z(R) = 0$. Therefore, $Q(R)$ is a non-singular R -module, which contains R as an essential cyclic submodule. Apply Theorem 2.1 and Proposition 3.1.

However, the previous result may fail if R is not a Goldie-ring.

Example 3.3. There exists a ring R without zero-divisors which has a right ideal which is not endomorphical.

Proof. Let R be a ring without zero-divisors which has infinite Goldie-dimension. Then, R contains a right ideal isomorphic to $\bigoplus_{\omega} R$. Let $\{x_n | n < \omega\}$ be an R -basis for $\bigoplus_{\omega} R$. It is easy to see that $\bigcup_{n < \omega} x_n R$ is an R -closed, strongly R -pure subset of $\bigoplus_{\omega} R$. Then, there is an R -homogeneous map $\phi: \bigoplus_{\omega} R \rightarrow \bigoplus_{\omega} R$ which is the identity on $\bigcup_{n < \omega} R$ and takes value 0 elsewhere. This shows that $\bigoplus_{\omega} R$ is not endomorphical.

The next result determines for which of the rings in Theorem 2.1 the class of non-singular modules which are rays for the class of non-singular modules coincides with the class of submodules of $Q(R)$.

Theorem 3.4. *The following conditions are equivalent for a ring R such that R is a semi-prime Goldie-ring:*

- (a) *Every non-singular module M which is an $E(M)$ -ray contains an essential locally cyclic submodule.*
- (b) *Every non-singular module M which is an $E(M)$ -ray has finite Goldie-dimension.*
- (c) *If M is a non-singular ray for the class of all non-singular modules, then M contains an essential locally cyclic submodule.*
- (d) *$Q(R) \cong \prod_{i=1}^n D_i$ where each D_i is a division algebra.*

Proof. (a) $\Rightarrow$ (b): Condition (a) implies that every non-singular R -module M which is an $E(M)$ -ray is isomorphic to a submodule of $Q(R)$, and therefore has finite Goldie-dimension.

(b) $\Rightarrow$ (d): Since R is a semi-prime Goldie-ring, we have

$$Q(R) \cong \prod_{i=1}^n S_i$$

where $S_i = \text{Mat}_{m_i}(D_i)$ for some division algebra D_i . Assume that $m_i > 1$ for some i . No generality is lost if we assume $i = 1$. We construct a non-singular module M of infinite Goldie-dimension which is an $E(M)$ -ray. The construction is based on a series of results which are established in the next paragraph.

We show that a $Q(R)$ -module M is an $E(M)$ -ray as an R -module if it is endomorphical as a $Q(R)$ -module. Suppose that $\phi: M \rightarrow M$ is an R -homogeneous map. If $r \in R$ and d is a regular element of R , then $\phi(x)r = \phi(xr) = \phi(xrd^{-1})d$ for all $x \in M$ yields that ϕ is $Q(R)$ -homogeneous. Therefore, ϕ is a homomorphism. Clearly, M is an $E(M)$ -ray once we have shown that it is an injective R -module. Since $Q(R)$ is semi-simple Artinian, it is enough to observe that $\bigoplus_J Q(R)$ is injective as an R -module for all index sets J . But this is an immediate consequence of [5, IX, Proposition 2.7].

Since $m_1 > 1$, we have that every S_1 -module is endomorphical by [1]. This holds in particular for $M = \bigoplus_{\omega} S_1$. We view M as an $Q(R)$ -module on which $S_2, \dots, S_n$ operate trivially. In particular, a map $\phi: M \rightarrow M$ is $Q(R)$ -homogeneous if and only if it is S_1 -homogeneous. Therefore, M is $Q(R)$ -endomorphical. By the results of the previous section, M is a non-singular R -module of infinite Goldie-dimension which is an $E(M)$ -ray. The resulting contradiction shows $m_i = 1$ for all $i = 1, \dots, n$.

(d) $\Rightarrow$ (a): Let M be a non-singular $E(M)$ -ray. Then, $E(M)$ is a $Q(R)$ -module and has a unique decomposition $M = \bigoplus_{i=1}^n M_i$ where M_i is a D_i -vector space on which D_j operates trivially for $j \neq i$. Suppose that $\phi: E(M) \rightarrow E(M)$ is a $Q(R)$ -homogeneous map; and choose $x, y \in E(M)$. There is a regular element $d \in R$ with $xd, yd \in M$. Since $\phi|_M$ is an R -homogeneous map from M to $E(M)$, we have $\phi(x+y)d = \phi(xd+yd) = \phi(xd) + \phi(yd) = (\phi(x) + \phi(y))d$. The fact that $E(M)$ is non-singular yields $\phi(x+y) = \phi(x) + \phi(y)$. Therefore, $E(M)$ is an endomorphical $Q(R)$ -module. By [3], each M_i is an endomorphical $Q(R)$ -module. Since D_j operates trivially on M_i if $j \neq i$, we obtain that M_i is an endomorphical D_i -module. Suppose that M_i has dimension at least 2 as a D_i -vector-space. Then, $V = D_i \oplus D_i$ is endomorphical. On the other hand, let U be the subspace of V which is generated by $(1, 0)$. If $0 \neq (x, y)d \in U$ for some $d \in D_i$, then $d \neq 0$ and $y = 0$. This shows that

U is strongly R -pure in V . By [3], the map $\phi: V \rightarrow V$ which is defined by

$$\phi(x) = \begin{cases} x & \text{if } x \in U, \\ 0 & \text{otherwise} \end{cases}$$

is an R -homogeneous map which is not additive. The resulting contradiction shows that each M_i has at most dimension 1 as a D_i -vector-space. Hence, $E(M)$ is a submodule of $Q(R)$. Now apply Theorem 2.1.

Since the proof of implication (a) $\Rightarrow$ (c) is trivial, it remains to show the converse in order to complete the proof: Let M be a non-singular $E(M)$ -ray. Suppose $M \not\subseteq Q(R)$, and consider a decomposition $E(M) = \bigoplus_{i=1}^n M_i$ into S_i -modules as in the proof of (b) $\Rightarrow$ (d). Assume (a) fails. By the equivalence of (a) and (d) which has been shown, we may assume that $M_1 \not\subseteq S_1$ and that all S_1 -modules are endomorphical. Let X be a non-singular R -module, and decompose $E(X) = \bigoplus_{i=1}^n X_i$ where each X_i is an S_i -module. If $\alpha: M_1 \rightarrow X$ is R -homogeneous, then $\alpha(M_1) \subseteq X_1$. Thus, α is an S_1 -homogeneous map from M_1 to X_1 . Observe that α induces an S_1 -homogenous map $\beta: M_1 \oplus X_1 \rightarrow M_1 \oplus X_1$ by $\beta(m, x) = (0, \alpha(m))$ for all $m \in M_1$ and $x \in X_1$. Since all S_1 -modules are endomorphical, β , and hence α , is additive. This shows that M_1 is a ray for the class of all non-singular modules. In particular, $M_1 \subseteq Q(R)$ by (c) and Theorem 2.1, which is not possible.

We want to remind the reader that condition (a) in the last result is equivalent to the requirement that every non-singular module M which is an $E(M)$ -ray is isomorphic to a submodule of $Q(R)$ by Theorem 2.1.

We now determine the semi-prime right Goldie-rings R for which all non-singular modules are endomorphical.

Theorem 3.5. *The following conditions are equivalent for a ring R such that R is a semi-prime right Goldie-ring:*

- (a) *Every non-singular R -module is a ray for the class of all non-singular modules.*
- (b) *Every non-singular R -module M is an $E(M)$ -ray.*
- (c) *Every non-singular R -module is endomorphical.*
- (d) *$Q(R) \cong \prod_{i=1}^n \text{Mat}_{m_i}(D_i)$ where D_i is a division algebra and $m_i > 1$ for $i = 1, \dots, n$.*

Proof. Observe that implications (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are obvious.

(c) $\Rightarrow$ (d): Since R is a semi-prime Goldie-ring, $Q(R) \cong \prod_{i=1}^n S_i$ where $S_i = \text{Mat}_{m_i}(D_i)$ for some division algebra D_i . If $m_i = 1$ for some i , then S_i itself is the only endomorphical S_i -module by Theorems 2.1 and 3.4. On the other hand, $S_i \times S_i$ is a non-singular R -module on which S_j operates trivially for all $j \neq i$. Therefore, $S_i \times S_i$ is endomorphical as an R -module. The resulting contradiction verifies (d).

(d) $\Rightarrow$ (a): By [1], we obtain that every $Q(R)$ -module is endomorphical. If M is a non-singular, injective R -module, then M can be viewed as a $Q(R)$ -module.

Let M and N be non-singular modules, and consider an R -homogenous map $\sigma: M \rightarrow N$. We show that σ extends to an R -homogeneous map $\tau: E(M) \rightarrow E(N)$. To define τ , let $x \in E(M)$, and choose a regular element $c \in R$ such that $xc \in M$. Since $E(N)$ is a non-singular $Q(R)$ -module, there is a unique $y \in E(N)$ with $\sigma(xc) = yc$. We claim that y is independent of the

chosen regular element c . If d is a regular element of R with $xd \in M$ and $z \in E(M)$ satisfies $\sigma(xd) = zd$, then there are $r, s \in R$ with s regular such that $c^{-1}d = rs^{-1}$. Since $ds = cr$ is regular, we obtain that $zds = \sigma(xd)s = \sigma(xds) = \sigma(xcr) = ycr$ implies $z = y$ if we observe that $E(N)$ is a non-singular R -module. If we set $\tau(x) = y$, then τ extends σ . It remains to show that τ is R -homogeneous.

If $x \in E(M)$ and $r \in R$, then there are regular elements $c, d \in R$ with $xc, xrd \in M$. Choose $s, t \in R$ with t regular such that $c^{-1}rd = st^{-1}$. The equation $rdt = cs$ yields $\tau(xr)dt = \sigma(xrdt) = \sigma(xcs) = \sigma(xc)s = \tau(x)cs$. Since dt is regular, $\tau(xr) = \tau(x)r$.

To show that σ is additive, it is enough to show this for τ . As in the proof of Theorem 3.4, we view τ as a $Q(R)$ -homogeneous map from $E(M)$ to $E(N)$. Define an R -homogeneous map $\lambda: E(M) \oplus E(N) \rightarrow E(M) \oplus E(N)$ by $\lambda(x, y) = (0, \tau(x))$. Since all $Q(R)$ -modules are endomorphical by [1, Theorem II.8], λ is additive, and the same holds for τ .

Corollary 3.6. *There exists a semi-prime right Goldie-ring R such that not all R -modules are endomorphical, although all non-singular R -modules are endomorphical.*

Proof. Let $R = \begin{bmatrix} \mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$. Since $Q \otimes_{\mathbb{Z}} R = \text{Mat}_2(\mathbb{Q})$ is the classical right ring of quotient of the semi-prime Goldie-ring R , every non-singular right R -module is endomorphical by the last theorem.

Assume that all R -modules are endomorphical. Observe that $I = \begin{bmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{bmatrix}$ is a two-sided ideal of R . It is easy to see that

$$R/I \cong \begin{bmatrix} \mathbb{Z}/2\mathbb{Z} & 0 \\ \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \end{bmatrix}.$$

If $N(R/I)$ denotes the nilradical of R/I , then $S = (R/I)/N(R/I) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is a ring epimorphic image of R . By [1, Theorem II.2], every S -module M is endomorphical. Since S is semi-simple Artinian, M is injective, and hence an $E(M)$ -ray. By Theorem 3.4, M is isomorphic to a submodule of $Q(S) = S$, which yields a contradiction.

It is furthermore possible to characterize the right non-singular rings R for which the non-singular endomorphical modules are precisely the non-singular modules of Goldie-dimension 1.

Theorem 3.7. *The following conditions are equivalent for a right non-singular ring R :*

- (a) R has Goldie-dimension at most 1.
- (b) The following conditions are equivalent for a non-singular right R -module M :
 - (i) M is endomorphical.
 - (ii) M is a ray for the class of all non-singular modules.
 - (iii) M has Goldie-dimension at most 1.

Proof. (a) $\Rightarrow$ (b): Observe that a right non-singular ring of right Goldie-dimension 1 has to be a semi-prime Goldie-ring. It remains to show that (i) $\Rightarrow$ (iii).

Consider a non-singular endomorphical R -module M , and assume that U and V are non-zero submodules of M with $U \cap V = 0$. Let $\tilde{U}$ and $\tilde{V}$ be the

$\mathcal{S}$ -closures of U and V in M respectively. Observe that $\tilde{U}/U = Z(M/U)$. Suppose that there are $x \in M$ and $r \in R$ such that $0 \neq xr \in \tilde{U}$. There is an essential right ideal I of R with $xrI \subseteq U$. If $s \in I$, then $xrs \in U$, and sR is an essential right ideal of R since R has right Goldie-dimension 1. Since R is a right non-singular ring, we have $rs \neq 0$. Therefore, rsR is an essential right ideal of R with $xrsR \subseteq U$. Therefore, $x \in \tilde{U}$.

By [3], the map $\phi: M \rightarrow M$ which is defined by

$$\phi(x) = \begin{cases} x & \text{if } x \in \tilde{U} \cup \tilde{V}, \\ 0 & \text{otherwise} \end{cases}$$

is R -homogeneous. Choose non-zero elements $a \in U$ and $b \in V$. If $a+b \in \tilde{U}$, then there is a non-zero $r \in R$ such that $(a+b)r \in U$. Then, $br \in U \cap V = 0$. Since rR is essential in R , we have $b \in Z(M) = 0$, which is not possible. Therefore, $a+b \notin \tilde{U} \cup \tilde{V}$. This shows

$$\phi(a+b) = 0 \neq a+b = \phi(a) + \phi(b).$$

Thus, M is not endomorphal, a contradiction.

(b) $\Rightarrow$ (a): Observe that R is a non-singular endomorphal module. Thus, R has Goldie-dimension at most 1 by (b).

Corollary 3.8. *Let R be a right non-singular ring of Goldie-dimension 1.*

- (a) *The class of endomorphal, non-singular R -modules is closed under submodules and essential extensions.*
- (b) *An endomorphal module M is either non-singular or satisfies $Z_2(M) = M$.*

Proof. (a) Observe that the class of non-singular modules of Goldie-dimension at most 1 is closed under submodules and essential extensions.

(b) Let M be an endomorphal module with $0 \neq Z(M) \neq M$. Suppose that $0 \neq xr \in Z(M)$ for some $x \in M$ and $r \in R$. There is an essential right ideal J of R with $xrJ = 0$. Since R has Goldie-dimension 1 and is non-singular, we have that the non-zero right ideal rJ of R is essential. Thus, $x \in Z(M)$. Therefore, the map $\phi: M \rightarrow M$ which is defined by

$$\phi(x) = \begin{cases} x & \text{if } x \in Z(M), \\ 0 & \text{otherwise} \end{cases}$$

is R -homogeneous. Choose $a \in M \setminus Z(M)$ and $0 \neq b \in Z(M)$. Suppose that there is an essential right ideal I of R with $(a+b)I = 0$. Without loss of generality, we can assume that I satisfies $bI = 0$ too. Then, $aI = 0$, which is not possible. Therefore, $\phi(a+b) = 0 \neq b = \phi(a) + \phi(b)$.

We conclude with a few results concerning singular modules.

Corollary 3.9. *Let R be a semi-prime right and left Goldie-ring. Then, $Q(R)/R$ is a $Q(R)/R$ -ray.*

Proof. By Theorem 2.2, $Q(R)$ is a locally cyclic R -module. Since the class of locally cyclic modules is closed under epimorphic images, the same holds for $Q(R)/R$. Now apply Proposition 3.1.

Corollary 3.10. *Let R be a semi-prime right hereditary right and left Goldie-ring. Every uniform injective module is endomorphal.*

Proof. Let M be a non-zero uniform injective module, and $\phi: R \rightarrow M$ a non-zero map. Since M is injective, ϕ extends to a map $\psi: Q(R) \rightarrow M$. The fact that M is uniform implies $\psi(Q(R))$ is an essential submodule of M . Moreover, since R is right hereditary, $\psi(Q(R))$ is injective too. This is only possible if $\psi(Q(R)) = M$. Since $Z(R) = 0$, we obtain that $Q(R)$ is locally cyclic by Theorem 2.2. The observation that the class of locally cyclic modules is closed with respect to homomorphic images yields that M is locally cyclic. In particular, M is endomorphal.

We want to point out that some of the results of this paper can be extended by considering conditions on the ring $R/Z_2(R)$ instead of assuming that R is right non-singular. However, since the categories of non-singular right R - and non-singular right $R/Z_2(R)$ -modules coincide, we have abstained from this generalization for the sake of a less complicated exposition.

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