ZERO ENTROPY AND DIRECTIONAL BERNOULLICITY OF A GAUSSIAN $\mathbb{Z}^2$ -ACTION

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ABSTRACT. We give an example of a Gaussian $\mathbb{Z}^2$ -action Φ with zero entropy which is weakly mixing, rigid and such that every non-trivial measure-preserving transformation Φ^g defined by Φ , $g \in \mathbb{Z}^2$, is a Bernoulli shift with infinite entropy.

1. INTRODUCTION

An important invariant in the ergodic theory of measure-preserving actions of the group $\mathbb{Z}^d$, $d \geq 1$, is the metric entropy. It has been defined by Kolmogorov in the case $d = 1$ ([6]) and by Conze, Katznelson, Weiss ([1], [5]) for $d \geq 2$. Conze has shown the following remarkable property of the entropy in the multidimensional case which has no analogue in the one-dimensional case. A measure-preserving $\mathbb{Z}^d$ -action Φ , $d \geq 2$, has zero entropy if some non-trivial transformation Φ^g defined by Φ , $g \in \mathbb{Z}^d$, has finite entropy. An interesting consequence of this property is a fact that one cannot smoothly realize $\mathbb{Z}^d$ -actions, $d \geq 2$, with positive entropy.

It is worth noting that Fried ([3]) has given a different definition of the entropy that do not degenerate in the smooth case.

Our example shows that the theorem converse to the above of Conze fails to be true and, additionally, all of the non-trivial transformations defined by the given $\mathbb{Z}^2$ -action are Bernoulli shifts and in particular they have completely positive entropy.

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2. PRELIMINARIES

Let $(X, \mathcal{B}, \mu)$ be a Lebesgue probability space. For a given sub- σ -algebra $\mathcal{A} \subset \mathcal{B}$ we denote by $L^2(\mathcal{A})$ the linear subspace of $L^2(X, \mu)$ consisting of $\mathcal{A}$ -measurable functions.

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Let Φ be a $\mathbb{Z}^d$ -action on $(X, \mathcal{B}, \mu)$, i.e. Φ is a homomorphism of $\mathbb{Z}^d$ into the group of all invertible measure-preserving transformations (automorphisms) of $(X, \mathcal{B}, \mu)$. The automorphism which corresponds to $a \in \mathbb{Z}^d$ is denoted by Φ^a . We denote by $h(\Phi)$ and $\Pi(\Phi)$ the entropy and the Pinsker σ -algebra of Φ respectively (cf. [1]). Let $\mathbb{T}^d$ be the d -dimensional torus and λ the Lebesgue measure on $\mathbb{T}^d$.

Let U be the unitary representation of $\mathbb{Z}^d$ on $L^2(X, \mu)$ defined by the formula

$$U^a f = f \circ \Phi^a, \quad a \in \mathbb{Z}^d,$$

and let E denote the spectral measure of U , i.e.

$$U^a = \int_{\mathbb{T}^d} z_1^{m_1} \cdots z_d^{m_d} E(dz_1 \cdots dz_d), \quad a = (m_1, \dots, m_d) \in \mathbb{Z}^d.$$

We associate, with every function $f \in L^2(X, \mu)$, the cyclic space $Z(f)$ and the spectral measure ν_f , i.e. $Z(f) = \overline{\text{Sp}}\{U^a f; a \in \mathbb{Z}^d\}$ and $\nu_f(A) = (E(A)f, f)$ where A is a Borel subset of $\mathbb{T}^d$.

We need in the sequel the following elementary properties of spectral measures.

Letting $f, g \in L^2(X, \mu)$ and $\alpha, \beta \in \mathbb{C}$,

$$(1) \quad \nu_{\alpha f + \beta g} \leq 2(|\alpha|^2 \nu_f + |\beta|^2 \nu_g).$$

$$(2) \quad \text{If the subspaces } Z(f) \text{ and } Z(g) \text{ are orthogonal, then } \nu_{\alpha f + \beta g} = |\alpha|^2 \nu_f + |\beta|^2 \nu_g,$$

$$(3) \quad |\nu_f - \nu_g| \leq (\|f\| + \|g\|) \cdot \|f - g\|.$$

From (3) easily follows

$$(4) \quad \text{If } f, f_n \in L^2(X, \mu), \nu_{f_n} \ll \nu \text{ where } \nu \text{ is a finite Borel measure on } \mathbb{T}^d, n \geq 1 \text{ and } \|f_n - f\| \rightarrow 0, \text{ then } \nu_f \ll \nu.$$

A $\mathbb{Z}^d$ -action Φ is said to be rigid if there exists a sequence $(a(n)) \subset \mathbb{Z}^d$ such that $\lim_{n \rightarrow \infty} a(n) = \infty$ and

$$\lim_{n \rightarrow \infty} \mu(\Phi^{a(n)} A \cap A) = \mu(A)$$

for every set $A \in \mathcal{B}$. Now we recall the definition of a Gaussian $\mathbb{Z}^d$ -action. Let X be the set of all real-valued functions defined on $\mathbb{Z}^d$, and let $\mathcal{B}$ denote the product σ -algebra of subsets of X .

Let $\xi_a: X \rightarrow \mathbb{R}$ be the projection on the a -th coordinate, i.e. $\xi_a(x) = x(a)$, $a \in \mathbb{Z}^d$. Let ρ be a finite symmetric measure on $\mathbb{T}^d$, and let μ be the probability measure on X such that the family $(\xi_a, a \in \mathbb{Z}^d)$ forms a stationary Gaussian process with the covariance function $R: \mathbb{Z}^d \rightarrow \mathbb{C}$ given by

$$R(a) = \int_{\mathbb{T}^d} z_1^{m_1} \cdots z_d^{m_d} \rho(dz_1 \cdots dz_d), \quad a = (m_1, \dots, m_d) \in \mathbb{Z}^d.$$

The $\mathbb{Z}^d$ -action Φ on $(X, \mathcal{B}, \mu)$ such that $(\Phi^a x)(b) = x(a+b)$, $a, b \in \mathbb{Z}^d$, is called the Gaussian $\mathbb{Z}^d$ -action with the spectral measure ρ . Using classical arguments (cf. [2]) one can show that a Gaussian $\mathbb{Z}^d$ -action Φ is weakly mixing iff ρ is continuous.

3. RESULT

The following result is an extension of the theorem of Pinsker ([8]).

Proposition. *Every Gaussian $\mathbb{Z}^d$ -action with a singular spectral measure has zero entropy.*

Proof. The idea of the proof is similar to that in the one-dimensional case ([7]). Therefore we give only a sketch of the proof. First we show that $\xi_o \in L^2(\Pi(\Phi))$ where o denotes the null vector of $\mathbb{Z}^d$. Let $\xi_o = f + g$ where $f \in L^2(\Pi(\Phi))$ and $g \in L^2(X, \mu) \ominus L^2(\Pi(\Phi))$. It follows from (2) that $\nu_{\xi_o} = \nu_f + \nu_g$. By Theorem 7 ([4]) the $\mathbb{Z}^d$ -action Φ has in the subspace $L^2(X, \mu) \ominus L^2(\Pi(\Phi))$ a countable Lebesgue spectrum. Therefore $\nu_g \ll \lambda$. But $\nu_g \ll \nu_{\xi_o} = \rho$ and ρ , by the assumption, is singular. Hence $\nu_g = 0$ and so $g = 0$. Thus $\xi_o = f \in L^2(\Pi(\Phi))$. Since the σ -algebra $\Pi(\Phi)$ is totally invariant, we have $\xi_a \in L^2(\Pi(\Phi))$, $a \in \mathbb{Z}^d$. Hence $\Pi(\Phi) = \mathcal{B}$, i.e. $h(\Phi) = 0$.

Theorem. *There exists a Gaussian $\mathbb{Z}^2$ -action Φ with zero entropy such that for every $a \in \mathbb{Z}^2 \setminus \{o\}$ the automorphism Φ^a is a Bernoulli shift with infinite entropy.*

Proof. Let $\alpha, \beta \in [0, 1)$ be irrational and independent over $\mathbb{Z}$. We consider the following mapping $\varphi: \mathbb{T} \rightarrow \mathbb{T}^2$ given by the formula

$$\varphi(e^{2\pi i t}) = (e^{2\pi i \alpha t}, e^{2\pi i \beta t}), \quad t \in [0, 1).$$

We consider a measure ρ on $\mathbb{T}^2$ defined as

$$\rho = \frac{1}{2}(\lambda\varphi^{-1} + \lambda(s \circ \varphi)^{-1}),$$

i.e.

$$\rho(A) = \frac{1}{2}[\lambda(\varphi^{-1}(A)) + \lambda(\varphi^{-1}(s^{-1}(A)))]$$

where $s(z, w) = (\bar{z}, \bar{w})$, $(z, w) \in \mathbb{T}^2$ and A is a Lebesgue measurable subset of $\mathbb{T}^2$. It is clear that ρ is a symmetric singular and continuous measure on $\mathbb{T}^2$. Let Φ be a Gaussian $\mathbb{Z}^2$ -action with the spectral measure ρ . By Proposition we have $h(\Phi) = 0$.

Since

$$R(a) = \frac{\sin 2\pi(\alpha m + \beta n)}{2\pi(\alpha m + \beta n)}, \quad a = (m, n) \in \mathbb{Z}^2,$$

we see that Φ is rigid.

Let $a = (p, q) \in \mathbb{Z}^2 \setminus \{o\}$. We claim that the automorphism Φ^a is a Bernoulli shift with $h(\Phi^a) = +\infty$. It is clear that Φ^a is a Gaussian automorphism ($\mathbb{Z}$ -action) on the space $(X, \mathcal{B}_a, \mu)$ where $\mathcal{B}_a$ denotes the σ -algebra generated by the projections ξ_{na} , $n \in \mathbb{Z}$. Now we check that the spectral measure of Φ^a is $\rho_a = \rho\psi_a^{-1}$ where $\psi_a(z, w) = z^p w^q$, $(z, w) \in \mathbb{T}^2$.

We have

$$\begin{aligned} \int_{\mathbb{T}} z^k \rho_a(dz) &= \int_X \xi_{ka} \cdot \xi_o d\mu = \int_{\mathbb{T}^2} z^{kp} w^{kq} \rho(dz dw) \\ &= \int_{\mathbb{T}^2} (\psi_a(z, w))^k \rho(dz dw) \\ &= \int_{\mathbb{T}} z^k (\rho\psi_a^{-1})(dz), \quad k \in \mathbb{Z}, \end{aligned}$$

and so $\rho_a = \rho\psi_a^{-1}$. Hence

$$\rho_a = \frac{1}{2}(\lambda(\psi_a \circ \varphi)^{-1} + \lambda(\psi_a \circ s \circ \varphi)^{-1}).$$

But

$$(\psi_a \circ \varphi)(e^{2\pi i t}) = e^{2\pi i(p\alpha + q\beta)t}$$

and

$$(\psi_a \circ s \circ \varphi)(e^{2\pi i t}) = e^{-2\pi i(p\alpha + q\beta)t}, \quad t \in [0, 1).$$

Since $p\alpha + q\beta \neq 0$, it is clear that ρ_a is a symmetric and absolutely continuous probability measure on $\mathbb{T}$. It follows from [8] that $h(\Phi^a) = +\infty$.

It remains to show that Φ^a is a Bernoulli shift. Let H be the Gaussian space generated by ξ_b , $b \in \mathbb{Z}^2$. By the standard spectral theory H is an orthogonal sum $H = \bigoplus_{k=1}^{\infty} Z(f_k)$ where $Z(f_k)$ is a cyclic space determined by f_k and U^a , $k \geq 1$.

It follows from elementary properties of Gaussian processes that every subspace $Z(f_k)$, $k \geq 1$, is a Gaussian one and the σ -algebra

$$\mathcal{B} = \sigma(\xi_b, b \in \mathbb{Z}^2)$$

is a product σ -algebra

$$\mathcal{B} = \prod_{k=1}^{\infty} \sigma(f, f \in Z(f_k)).$$

It is clear that for every $b \in \mathbb{Z}^2$ the spectral measure of ξ_b is absolutely continuous. Hence, applying (1) and (4), we see that the spectral measures of f_k , $k \geq 1$, are also absolutely continuous. It follows from [9] that the factor of Φ^a determined by the σ -algebra $\sigma(f; f \in Z(f_k))$, $k \geq 1$, is a Bernoulli shift. Therefore Φ^a is also a Bernoulli shift that completes the proof.

Our example shows that there are Z^2 -actions with a great difference between two-dimensional mixing properties (rigidity) and mixing properties of automorphisms generating these Z^2 -actions (Bernoullicity).

We would like to thank the referee for an example of a strongly mixing Z^2 -action Φ with zero entropy and such that every automorphism Φ^g , $g \in \mathbb{Z}^2 \setminus \{o\}$, is Bernoulli with finite entropy. While our example uses Gaussian processes, the referee's example is of a purely algebraic nature.

REFERENCES

1. J. P. Conze, *Entropie d'un groupe abélien de transformations*, Z. Wahrsch. Verw. Gebiete **25** (1972), 11–30.
2. I. P. Cornfeld, S. V. Fomin, and Y. G. Sinai, *Ergodic theory*, Springer-Verlag, Berlin and New York, 1982.
3. D. Fried, *Entropy for smooth abelian actions*, Proc. Amer. Math. Soc. **87** (1983), 111–116.
4. B. Kamiński, *The theory of invariant partitions for Z^d -actions*, Bull. Polish Acad. Sci. Math. **29** (1981), 349–362.
5. Y. Katznelson and B. Weiss, *Commuting measure preserving transformations*, Israel J. Math. **12** (1972), 161–173.
6. A. N. Kolmogorov, *A new metric invariant of transient dynamical systems and automorphisms of Lebesgue spaces*, Dokl. Akad. Sci. SSSR **119** (1958), 861–864.
7. W. Parry, *Topics in ergodic theory*, Cambridge Univ. Press, London and New York, 1981.

8. M. S. Pinsker, *Dynamical systems with completely positive and zero entropy*, Dokl. Akad. Nauk SSSR **133** (1960), 1025–1026.
9. M. Smorodinsky, *A partition on a Bernoulli shift which is not weakly Bernoulli*, Math. Systems Theory **5** (1971), 201–203.

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