NOETHERIAN DOMAINS WITH MANY MORE ELEMENTS THAN HEIGHT-ONE PRIMES

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ABSTRACT. Extending previous results of L. Claborn and H.W. Lenstra, Jr., we show that if D is a Krull domain with a set of height-one primes $X^{(1)}$ that satisfies either (1) D contains a subset k with $|k| > |X^{(1)}|$ and for $\mu \neq \lambda \in k$, $\mu - \lambda$ is a unit, or (2) $|D| > |X^{(1)}|^{\aleph_0}$, then D is a Euclidean domain. We also show that any Noetherian ring satisfying (1) or Noetherian domain satisfying (2) has Krull dimension at most one.

It is a well-known consequence of the Principal Ideal Theorem that a Noetherian domain (or ring) with only a finite number of height-one primes has Krull dimension at most one. (For example, see the proof of [5, Theorem 144].) It is also well known that a Dedekind domain (even a Krull domain) with a finite number of height-one primes is a semilocal PID. Moreover, Estes and Ohm [3, Corollary 5.4] and Samuel [7, Proposition 5] showed that a semilocal PID is a Euclidean domain with (smallest) algorithm $\phi(x) = \sum_{P \in X^{(1)}} v_P(x)$. Here for any ring R , $X^{(1)} = X^{(1)}(R)$ is the set of height-one prime ideals of R , and for a Krull domain D and $P \in X^{(1)}(D)$, v_P is the normalized discrete valuation associated with P . And we define an integral domain D to be a *Euclidean domain* if there exists an ordinal number W and a function $\phi: D - \{0\} \rightarrow W$ satisfying the condition: for $a, b \in D$ with $b \neq 0$, there exist $q, r \in D$ with $a = bq + r$ and either $r = 0$ or $\phi(r) < \phi(b)$. Then ϕ is called the *algorithm* for D . Claborn [2] (or see [4, Corollary 13.10]) “generalized” this result concerning Dedekind domains by showing that if D is a Dedekind domain with $|D| > |X^{(1)}|^{\aleph_0}$, then D is a PID. (However, Claborn remarks that a Dedekind domain D with $|D| > |X^{(1)}|$ need not be a PID. For an example of such a Dedekind domain, see [4, Example 15.16].) Then Lenstra [6] showed that such a Dedekind domain is actually a Euclidean domain with algorithm $\phi: D - \{0\} \rightarrow \omega^2$ where $\phi(x) = \omega \cdot \sum_{P \in Z} v_P(x) + \sum_{P \in X^{(1)} - Z} v_P(x)$ for some finite subset $Z \subseteq X^{(1)}$. We show that Lenstra’s result (and hence Claborn’s result) holds for an arbitrary Krull domain. We also show that for any Noetherian domain D with $|D| > |X^{(1)}|^{\aleph_0}$, $\dim D \leq 1$. (We are allowing the possibility that D is a field.)

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But first we generalize another result of Claborn and Lenstra. Claborn [2] (or see [4, Proposition 13.7]) proved that if a Dedekind domain D contains a subfield k with $|D| = |k| > |X^{(1)}|$, then D is a PID. Lenstra [6] sharpened Claborn's result by assuming only that k is a subset of the Dedekind domain D with $|k| > |X^{(1)}|$ and $\lambda - \mu \in U(D) \cup \{0\}$ ($U(D)$ is the group of units of D) for all $\lambda, \mu \in k$ and then concluding that D is a Euclidean domain with algorithm $\phi(x) = \sum_{P \in X^{(1)}} v_P(x)$. Our first theorem extends this to arbitrary Krull domains.

Lemma 1. *Let D be a Krull domain. Then the cardinality of the set of principal ideals of D generated by irreducible elements is $|X^{(1)}|$. If $X^{(1)}$ is infinite, the cardinality of the set of principal ideals of D is also equal to $|X^{(1)}|$.*

Proof. Let Irr be the set of principal ideals of D generated by irreducible elements and let $P(D)_+$ be the set of nonzero (integral) principal ideals of D . First, suppose that $X^{(1)}$ is finite. Then D is a semilocal PID with $X^{(1)} = Irr$. Next, suppose that $X^{(1)}$ is infinite. Since each $P \in X^{(1)}$ contains an element of Irr and each element of Irr is contained in only finitely many height-one primes, $|X^{(1)}| \leq |Irr|$. So $|X^{(1)}| \leq |Irr| \leq |P(D)_+|$. Now two nonzero principal ideals (a) and (b) of D are equal if and only if $v_P(a) = v_P(b)$ for each $P \in X^{(1)}$. Since the set of functions from $X^{(1)}$ to $\mathbb{N}$ which are zero almost everywhere has cardinality $|X^{(1)}|$, we have $|P(D)_+| \leq |X^{(1)}|$.

Lemma 1 may also be interpreted in terms of the multiplicative group $P(D)$ of nonzero principal fractional ideals of D (called the group of divisibility of D). Since $P(D)$ is isomorphic to a subgroup of the free abelian group of all functions from $X^{(1)}$ to $\mathbb{Z}$ which are zero almost everywhere, $P(D)$ is itself free. By Lemma 1, $\text{rank } P(D) = |X^{(1)}|$ and if $X^{(1)}$ is infinite, $\text{rank } P(D) = |P(D)| = |Irr|$.

Theorem 2. *Let D be a Krull domain that contains a subset k (e.g., a subfield k) with $|k| > |X^{(1)}|$ and $\lambda - \mu \in U(D) \cup \{0\}$ for all $\lambda, \mu \in k$. Then D is Euclidean with smallest algorithm $\phi(x) = \sum_{P \in X^{(1)}} v_P(x)$.*

Proof. As previously mentioned, the case where $X^{(1)}$ is finite has already been handled in [3] and [7]. So suppose that $X^{(1)}$ is infinite. Let $a, b \in D$ with $b \neq 0$ and $a \notin (b)$. By Lemma 1, there exist $\lambda \neq \mu \in k$ with $(a + \lambda b) = (a + \mu b)$. Then $(a + \lambda b) = (a, b)$, so $(a + \lambda b) = (a, b)_v$, where for a nonzero ideal I , $I_v = (I^{-1})^{-1}$. Then $v_P(a + \lambda b) = \min\{v_P(a), v_P(b)\} \leq v_P(b)$ for all $P \in X^{(1)}$ with strict inequality for at least one P . Hence the element $r = a + \lambda b$ has $\phi(r) < \phi(b)$ and $a = b(-\lambda) + r$. Thus ϕ is an algorithm. By [7, Corollary 2] ϕ is the smallest algorithm.

Corollary 3. *Let R be a Noetherian ring that contains a subset k (e.g., a subfield) with the properties that $|k| > |X^{(1)}|$ and $\lambda - \mu \in U(R) \cup \{0\}$ for all $\lambda, \mu \in k$. Then for each minimal prime ideal P of R , the integral closure $\bar{R}/\bar{P}$ of R/P is a Euclidean domain. Hence $\dim R \leq 1$.*

Proof. Let P be a minimal prime ideal of R . Then R/P is a Noetherian domain with $X^{(1)}(R/P) \leq X^{(1)}(R) < |k| = |\bar{k}|$ where $\bar{k} = \{\mu + P \mid \mu \in k\}$. Thus R/P satisfies the hypothesis of the corollary, so we may assume that R is a Noetherian integral domain. If $|X^{(1)}(R)| < \infty$, then $|X^{(1)}(\bar{R})| < \infty$ and hence $\bar{R}$ is a semilocal PID and thus a Euclidean domain with $\dim R = \dim \bar{R} \leq 1$.

Suppose that $X^{(1)}(R)$ is infinite. Then $|k| > |X^{(1)}(R)| = |X^{(1)}(\bar{R})|$ and $\bar{R}$ is a Krull domain (the equality and the fact that $\bar{R}$ is a Krull domain is part of the Mori-Nagata Integral Closure Theorem [4, Theorem 4.3]). By Theorem 2, $\bar{R}$ is a Euclidean domain. Hence $\dim R = \dim \bar{R} \leq 1$.

Example 4. Note that for any pair of cardinal numbers $\alpha \leq \beta$ with β infinite, there is a Euclidean domain D with subfield k satisfying $|k| = |D| = \beta$ and $|X^{(1)}(D)| = \alpha$. For let k be a field with $|k| = \beta$ and $\{f_j\}_{j \in \Delta}$ a set of nonassociate irreducible polynomials of $k[X]$ with $|\Delta| = \alpha$. Then $D = k[X]_S$ where $S = k[X] - \bigcup (f_j)$ has the desired properties. If $\alpha < \beta$ we have seen that $\phi(x) = \sum v_P(x)$ is an algorithm for D . However, if $\alpha = \beta$, ϕ may or may not be an algorithm for D . The smallest algorithm for $k[X]$ is $\theta(f) = \deg f$ [7, page 290] and we always have $\phi(f) \leq \theta(f)$, so ϕ is an algorithm if and only if every irreducible polynomial of $k[X]$ has degree one, that is, k is algebraically closed. An example where the set k in Theorem 1 is not a field is as follows.

Let $\{X_\lambda \mid \lambda \in \Lambda\}$ be a set of indeterminates over $\mathbb{Z}$ and let $D_\Lambda = \mathbb{Z}[\{X_\lambda\}] = \mathbb{Z}[\{X_\lambda\}]_S$ where $S = \{f \in \mathbb{Z}[\{X_\lambda\}] \mid \text{the coefficients of } f \text{ generate } \mathbb{Z}\}$. Then D_Λ is a Euclidean domain with $X^{(1)}(D_\Lambda) = \text{Spec}(D_\Lambda) - \{0\} = \{pD_\Lambda \mid p \text{ is a nonzero prime of } \mathbb{Z}\}$ (see, for example [3]). Take $k = \{X_\lambda \mid \lambda \in \Lambda\}$, so $|k| = |\Lambda|$ and for $X_\lambda, X_\mu \in k$, $X_\lambda - X_\mu \in U(D_\Lambda) \cup \{0\}$. Certainly $|k|$ may be made arbitrarily large.

Lemma 5. *Let D be an integral domain containing a prime ideal Q with $\text{ht } Q > 1$ satisfying the following condition: if $0 \neq a \in Q$, then there are only finitely many prime ideals $P \subseteq Q$ that are minimal over (a) and any such prime ideal P has height one. Let $X^{(1)}(Q) = \{P \in X^{(1)}(D) \mid P \subseteq Q\}$. Then $|D| \leq |X^{(1)}(Q)|^{\aleph_0}$.*

Proof. Let b_1, b_2 be distinct nonzero elements of Q . Let $Q_1, \dots, Q_S$ be the elements of $X^{(1)}(Q)$ containing $b_2 - b_1$. Since each Q/Q_i is infinite, by [1, Chapter 2, §1, exercise 1], there exists $b_3 \in Q - \bigcup_{i=1}^S (b_2 + Q_i)$. So $b_3 - b_2$ is not contained in any of the height-one primes contained in Q that contain $b_2 - b_1$. In a similar manner, we inductively choose an element $b_n \in Q$ ($n > 3$) such that for $i < j < n$, $b_n - b_j$ is not contained in any element of $X^{(1)}(Q)$ containing $b_j - b_i$. Let $a \in Q$. For each $n \geq 1$, choose $P_n \in X^{(1)}(Q)$ with $a - b_n \in P_n$. Note that $\{P_n\}$ is infinite. For if $\{P_n\}$ is finite, there exists $i < j < k$ with $P_i = P_j = P_k$. But then $b_k - b_j, b_j - b_i \in P_k$, a contradiction. Define the function $S: Q \rightarrow (X^{(1)}(Q))^{\aleph_0}$ by $S(a) = (P_n)$. Now S is injective. For if $a, b \in Q$ with $S(a) = S(b)$, then $a - b = (a - b_n) - (b - b_n) \in P_n$ for each $n \geq 1$. Since a nonzero element of Q is contained in only finitely many height-one primes, we must have $a = b$. Thus $|Q| \leq |X^{(1)}(Q)|^{\aleph_0}$. Since $|D| = |Q|$, the result follows.

Theorem 6. *Let D be an integral domain satisfying the following two conditions for each maximal ideal M of D : (i) every prime ideal P of D_M minimal over a nonzero proper principal ideal of D_M has $\text{ht } P = 1$ and (ii) every nonzero proper principal ideal of D_M is contained in only finitely many height-one prime ideals of D_M . (For example, these two conditions hold if D_M is a Krull domain or Noetherian domain for each maximal ideal M of D .) Then $|X^{(1)}(D)|^{\aleph_0} < |D|$ implies $\dim D \leq 1$.*

Proof. Suppose that $\dim D > 1$; so D has a maximal ideal Q with $\text{ht } Q > 1$. Note that Q satisfies the hypothesis for the prime ideal Q in Lemma 5. Hence

by Lemma 5, $|D| \leq |X^{(1)}(Q)|^{\aleph_0} \leq |X^{(1)}(D)|^{\aleph_0}$, a contradiction. Thus $\dim D \leq 1$.

Theorem 7. *Let D be a Krull domain with $|D| > |X^{(1)}|^{\aleph_0}$. Then D is a Euclidean domain with algorithm $\phi : D - \{0\} \rightarrow \omega^2$ given by $\phi(x) = \omega \cdot \sum_{P \in Z} v_P(x) + \sum_{P \in X^{(1)} - Z} v_P(x)$ for some finite subset $Z \subseteq X^{(1)}$. Hence $\dim D \leq 1$.*

Proof. By Theorem 6, $\dim D \leq 1$ and hence D is a Dedekind domain. The result now follows from the previously mentioned result of Lenstra [6].

Corollary 8. *Let D be a Noetherian domain with $|D| > |X^{(1)}|^{\aleph_0}$. Then the integral closure $\bar{D}$ of D is a Euclidean domain and hence $\dim D \leq 1$.*

Proof. If $|X^{(1)}(D)| < \infty$, then D is a one-dimensional semilocal domain and hence $\bar{D}$ is a semilocal PID and hence a Euclidean domain. So we may assume that $X^{(1)}(D)$ is infinite. As remarked in Corollary 3, $\bar{D}$ is a Krull domain with $|\bar{D}| = |D| > |X^{(1)}(D)|^{\aleph_0} = |X^{(1)}(\bar{D})|^{\aleph_0}$. Thus $\bar{D}$ is a Euclidean domain and hence $\dim D = \dim \bar{D} \leq 1$.

Corollary 8, unlike Corollary 3, cannot be extended to Noetherian rings with zero divisors. Let K be a field with $|K| > \aleph_0^{\aleph_0}$. Then $R = K \times \mathbb{Q}[X_1, \dots, X_n]$ is an n -dimensional integrally closed Noetherian ring with $|R| = |K| > \aleph_0^{\aleph_0} = |X^{(1)}(R)|^{\aleph_0}$. Here R has two minimal prime ideals. However, Corollary 8 does carry over to Noetherian rings with a unique minimal prime ideal.

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