ON THE C^∞ WAVE-FRONT SET OF SOLUTIONS OF FIRST-ORDER NONLINEAR PDE'S

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ABSTRACT. Let $\Omega \subset \mathbb{R}^{m+1}$ be a neighborhood of the origin and assume $u \in C^2(\Omega)$ is a solution of the nonlinear PDE

$$u_t = f(x, t, u, u_x),$$

where $f(x, t, \zeta_0, \zeta)$ is C^∞ in the variables $(x, t) \in \mathbb{R}^m \times \mathbb{R}$ and holomorphic in the variables $(\zeta_0, \zeta) \in \mathbb{C} \times \mathbb{C}^m$. We present a proof that

$$WF(u) \subset \text{char}(L^u),$$

where $WF(u)$ denotes the C^∞ wave-front set of u and $\text{char}(L^u)$ is the characteristic set of the linearized operator

$$L^u = \frac{\partial}{\partial t} - \sum_{j=1}^m \left(\frac{\partial f}{\partial \zeta_j} \right) (x, t, u, u_x) \frac{\partial}{\partial x_j}.$$

1. INTRODUCTION

Let u be a C^2 solution of the nonlinear PDE

$$u_t = f(x, t, u, u_x)$$

where $f(x, t, \zeta_0, \zeta)$ is C^∞ in the variables $(x, t) \in \mathbb{R}^m \times \mathbb{R}$ and holomorphic in the variables $(\zeta_0, \zeta) \in \mathbb{C} \times \mathbb{C}^m$. Here (x, t, ζ_0, ζ) varies in an open set.

In this article we present a simple proof of a result found in Chemin [1] that, under the hypothesis above, the C^∞ wave-front set of u is contained in the characteristic set of the linearized operator

$$L^u = \frac{\partial}{\partial t} - \sum_{j=1}^m \left(\frac{\partial f}{\partial \zeta_j} \right) (x, t, u, u_x) \frac{\partial}{\partial x_j}.$$

In Hanges and Treves [2] it is shown that under the additional hypothesis that f be analytic in the variables (x, t) the analytic wave-front set of u is contained

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in the characteristic set of the linearized operator L^u . Our proof follows the general lines of Hanges and Treves [2] and can be used to simplify their argument as well.

2. PRELIMINARIES

Let $\Omega \subseteq \mathbb{R}_x^m \times \mathbb{R}_t$ be a neighborhood of the origin and consider the complex vector field defined on Ω

$$L = \frac{\partial}{\partial t} + \sum_{j=1}^m a_j(x, t) \frac{\partial}{\partial x_j},$$

where $a_j \in C^1(\Omega)$ for $j = 1, \dots, m$.

Suppose that there exist C^1 functions $\Psi_1(x, t), \dots, \Psi_m(x, t)$ defined on Ω such that if we write $\Psi = (\Psi_1, \dots, \Psi_m)$ and $Z(x, t) = x + t\Psi(x, t)$ then Z is an approximate solution of $LZ = 0$ in the sense that LZ is t -flat at $t = 0$, i.e.,

$$(1) \quad \forall \nu \in \mathbb{N} \exists C_\nu > 0: \quad |LZ(x, t)| \leq C_\nu |t|^\nu, \quad \forall (x, t) \in \Omega$$

(cf. [5], Chapter X, Definition 2.2, page 528).

Now consider $U \subset \subset \Omega$, a neighborhood of the origin where $Z_x(x, t)$ is non-singular. Observe that

$$LZ(x, 0) = Z_t(x, 0) + a(x, 0) = 0$$

and if we define $b = -Z_x^{-1}Z_t$ and

$$(2) \quad L_1 = \frac{\partial}{\partial t} + \sum_{j=1}^m b_j(x, t) \frac{\partial}{\partial x_j}$$

then L_1 is a complex vector field defined on U with the property

$$L_1 Z = Z_t + Z_x b = 0.$$

Note that

$$b(x, 0) = -Z_t(x, 0) = a(x, 0)$$

so that when $t = 0$ the vector fields L and L_1 coincide. In particular, the respective characteristic sets also coincide for $t = 0$. The symbol of L_1 is given by

$$\sigma(x, t, \xi, \tau) = \tau - Z_t(x, t) \cdot ({}^t Z_x^{-1}(x, t)\xi).$$

Then, for $t = 0$, the characteristic set of L_1 (and of L) is given by the equation

$$\sigma(x, 0, \xi, \tau) = \tau - \Psi(x, 0) \cdot \xi = 0$$

or equivalently

$$\begin{cases} \tau &= \operatorname{Re} \Psi(x, 0) \cdot \xi, \\ 0 &= \operatorname{Im} \Psi(x, 0) \cdot \xi. \end{cases}$$

Suppose now that $h \in C^1(U)$ is an approximate solution of $Lh = 0$ in the sense of (1). We have

$$LZ = LZ - L_1 Z = Z_x(a - b).$$

Thus

$$L_1 h = Lh + (L_1 - L)h = Lh + (b - a) \cdot h_x = Lh - (Z_x^{-1} LZ) \cdot h_x.$$

Since Lh and LZ are t -flat at $t = 0$, it follows that $L_1 h$ is also t -flat at $t = 0$ on U .

Consider now an open ball $B \subset \mathbb{R}^m$ and an interval $I \subset \mathbb{R}$, both centered at the origin and small enough so that $\overline{B \times I} \subset U$. Look at the FBI transform of h

$$\mathcal{F}h(t; z, \zeta) = \int_B e^{i\zeta \cdot (z - Z(x, t)) - \langle \zeta \rangle (z - Z(x, t))^2} h(x, t) d_x Z(x, t)$$

with the notation $\langle \zeta \rangle = \sqrt{\zeta \cdot \zeta}$, using the main branch of the square root, and $d_x Z(x, t) = \det Z_x(x, t) dx$. Here we have $\zeta \in \mathbb{C}^m$ with $|\operatorname{Re} \zeta| > |\operatorname{Im} \zeta|$. Observe that

$$\mathcal{F}h(0; z, \zeta) = \int_B e^{i\zeta \cdot (z - x) - \langle \zeta \rangle (z - x)^2} h(x, 0) dx.$$

Now we are all set to state the following lemma.

Lemma 2.1. *Suppose there exist $\Psi_1, \dots, \Psi_m \in C^1(\Omega)$ such that $Z = x + t\Psi(x, t)$ is an approximate solution of $LZ = 0$ in the sense of (1). Let $\xi^0 \in \mathbb{R}^m \setminus \{0\}$ be such that $\xi^0 \cdot \operatorname{Im} \Psi(0, 0) < 0$. Then there is a conic neighborhood Γ of ξ^0 in $\mathbb{R}^m \setminus \{0\}$ and a neighborhood $\mathcal{O} \subset \mathbb{C}^m$ of the origin such that $\mathcal{F}h(0; z, \xi)$ is rapidly decreasing in $\xi \in \Gamma$, for all approximate solutions $h \in C^1(\Omega)$ of $Lh = 0$ in the sense of (1) and all $z \in \mathcal{O}$.*

Proof. If $H(x, t)$ is a C^1 function defined on U , then

$$d(H dZ) = (L_1 H) dt \wedge dZ.$$

Thus if $0 < t \in I$, then

$$\begin{aligned} \int_{B \times [0, t]} (L_1 H) dt \wedge dZ &= \int_{\partial(B \times [0, t])} H dZ \\ &= \int_{x \in B} H(x, t) d_x Z(x, t) - \int_{x \in B} H(x, 0) d_x Z(x, 0) \\ &\quad + \int_{(x, s) \in \partial B \times [0, t]} H(x, s) \varphi(x, s) ds, \end{aligned}$$

for some continuous $(m - 1)$ -form φ . We will apply this equation to

$$H(x, t) = e^{i\zeta \cdot (z - Z(x, t)) - \langle \zeta \rangle (z - Z(x, t))^2} h(x, t)$$

with z and ζ acting as parameters. We have the following inequality:

$$(3) \quad |\mathcal{F}h(0; z, \zeta)| \leq \left| \int_0^t \int_{x \in B} e^{i\zeta \cdot (z - Z(x, s)) - \langle \zeta \rangle \langle z - Z(x, s) \rangle^2} L_1 h(x, s) d_x Z(x, s) ds \right| \\ + \left| \int_{x \in B} e^{i\zeta \cdot (z - Z(x, t)) - \langle \zeta \rangle \langle z - Z(x, t) \rangle^2} h(x, t) d_x Z(x, t) \right| \\ + \left| \int_0^t \int_{x \in \partial B} e^{i\zeta \cdot (z - Z(x, s)) - \langle \zeta \rangle \langle z - Z(x, s) \rangle^2} h(x, s) \varphi(x, s) ds \right|.$$

Write

$$Q(x, t, z, \zeta) = i\zeta \cdot (z - Z(x, t)) - \langle \zeta \rangle \langle z - Z(x, t) \rangle^2$$

and suppose for the moment that $z = 0$ and $\zeta = \xi \in \mathbb{R}^m \setminus \{0\}$. Then

$$Q(x, t, 0, \xi) = -i\xi \cdot Z(x, t) - |\xi| \langle Z(x, t) \rangle^2 \\ = -i\xi \cdot (x + t\Psi(x, t)) - |\xi| \langle x + t\Psi(x, t) \rangle^2$$

and

$$\operatorname{Re} Q(x, t, 0, \xi) = t\xi \cdot \operatorname{Im} \Psi(x, t) - |\xi| \operatorname{Re} \langle x + t\Psi(x, t) \rangle^2.$$

Note that

$$\langle x + t\Psi(x, t) \rangle^2 = |x|^2 + t^2 \langle \Psi(x, t) \rangle^2 + 2tx \cdot \Psi(x, t),$$

so

$$\operatorname{Re} \langle x + t\Psi(x, t) \rangle^2 = |x|^2 + t^2 |\operatorname{Re} \Psi(x, t)|^2 - t^2 |\operatorname{Im} \Psi(x, t)|^2 \\ + 2tx \cdot \operatorname{Re} \Psi(x, t).$$

Hence

$$(4) \quad \operatorname{Re} Q(x, t, 0, \xi) = t\xi \cdot \operatorname{Im} \Psi(x, t) - |\xi| \left[|x|^2 + t^2 |\operatorname{Re} \Psi(x, t)|^2 \right. \\ \left. - t^2 |\operatorname{Im} \Psi(x, t)|^2 + 2tx \cdot \operatorname{Re} \Psi(x, t) \right].$$

Since Ψ is of class C^1 on $\overline{B \times I}$ there exists a constant $M > 0$ such that

$$|\Psi(0, 0) - \Psi(x, t)| \leq M(|x| + |t|) \quad \forall (x, t) \in \overline{B \times I}.$$

Hence

$$t\xi \cdot \operatorname{Im} \Psi(x, t) \leq t\xi \cdot \operatorname{Im} \Psi(0, 0) + Mt|\xi|(|x| + |t|).$$

Note that

$$Mt|x| \leq \frac{1}{2} \left(\varepsilon^2 |x|^2 + \frac{1}{\varepsilon^2} M^2 t^2 \right)$$

and

$$|2tx \cdot \operatorname{Re} \Psi(x, t)| \leq \varepsilon^2 |x|^2 + \frac{t^2}{\varepsilon^2} |\operatorname{Re} \Psi(x, t)|^2.$$

So, from (4), there exists $K > 0$ such that

$$\operatorname{Re} Q(x, t, 0, \xi) \leq t\xi \cdot \operatorname{Im} \Psi(0, 0) + |\xi| t^2 K, \quad \forall (x, t) \in \overline{B \times I}.$$

Thus if $C_0 = -\xi^0 \cdot \text{Im } \Psi(0, 0) > 0$, then there is a conic neighborhood Γ of ξ^0 in $\mathbb{R}^m \setminus \{0\}$, a neighborhood $\mathcal{O} \subset \mathbb{C}^m$ of the origin and $\delta > 0$ such that if $\xi \in \Gamma$, $z \in \mathcal{O}$ and $0 < t < \delta$ then

$$\text{Re } Q(x, t, z, \xi) \leq -C_0 t |\xi|/2.$$

So we can estimate

$$\begin{aligned} & \left| |\xi|^\nu \int_0^t \int_{x \in B} e^{i\xi \cdot (z - Z(x, s)) - |\xi|(z - Z(x, s))^2} L_1 h(x, s) d_x Z(x, s) ds \right| \\ & \leq \int_0^t \int_{x \in B} e^{-C_0 s |\xi|/2} |L_1 h(x, s)| |\xi|^\nu |\det Z_x(x, s)| dx ds \\ & \leq \text{const} \int_0^t \int_{x \in B} e^{-C_0 s |\xi|/2} s^{\nu-1} |\xi|^\nu |\det Z_x(x, s)| dx ds \\ & \leq \text{const} \int_0^\infty e^{-C_0 \lambda/2} \lambda^{\nu-1} d\lambda \quad (\lambda = s|\xi|) \end{aligned}$$

for all $\xi \in \Gamma$ and $z \in \mathcal{O}$. Also observe that

$$\left| \int_{x \in B} e^{i\xi \cdot (z - Z(x, t)) - |\xi|(z - Z(x, t))^2} h(x, t) d_x Z(x, t) \right| \leq \text{const } e^{-C_0 t |\xi|/2}$$

has exponential decay in $\xi \in \Gamma$.

For the last term in (3) we have the additional information that $x \in \partial B$. From (4) we have that if $t > 0$ is small enough then, shrinking $\mathcal{O}$ if necessary,

$$\text{Re } Q(x, t, z, \xi) \leq -|\xi| r^2/2$$

where r is the radius of B , $z \in \mathcal{O}$ and $\xi \in \mathbb{R}^m \setminus \{0\}$. Hence

$$\left| \int_0^t \int_{x \in \partial B} e^{i\xi \cdot (z - Z(x, s)) - |\xi|(z - Z(x, s))^2} h(x, s) \varphi(x, s) ds \right| \leq \text{const } e^{-|\xi| r^2/2}$$

has exponential decay in any direction $\xi \in \mathbb{R}^m \setminus \{0\}$. $\square$

Consider now the vector field

$$L^\theta = \frac{\partial}{\partial s} - e^{-i\theta} L,$$

where $s \in \mathbb{R}$ is a new variable and $\theta \in [0, 2\pi)$ is a parameter. Suppose that for all $\theta \in [0, 2\pi)$ there exist C^1 functions $\Psi_1^\theta(x, t, s), \dots, \Psi_m^\theta(x, t, s)$ defined on $\Omega \times J$ ($J \subset \mathbb{R}$ is an open interval centered at the origin) so that

$$Z_j^\theta(x, t, s) = x_j + s \Psi_j^\theta(x, t, s), \quad j = 1, \dots, m$$

are approximate solutions of $L^\theta Z_j^\theta = 0$ in the sense that $L^\theta Z_j^\theta$ ($j = 1, \dots, m$) are s -flat at $s = 0$. Define also $\Psi_{m+1}^\theta(x, t, s) = e^{-i\theta}$ and $Z_{m+1}^\theta(x, t, s) = t + e^{-i\theta} s$ and note that $L^\theta Z_{m+1}^\theta = 0$. If we write $\Psi^\theta = (\Psi_1^\theta, \dots, \Psi_{m+1}^\theta)$ and $Z^\theta = (Z_1^\theta, \dots, Z_{m+1}^\theta)$, then

$$Z_s^\theta(0, 0, 0) = \Psi^\theta(0, 0, 0) = - \begin{pmatrix} e^{-i\theta} a(0, 0) \\ e^{-i\theta} \end{pmatrix} = e^{-i\theta} \begin{pmatrix} \Psi(0, 0) \\ -1 \end{pmatrix}$$

and

$$\begin{aligned} \begin{pmatrix} \xi \\ \tau \end{pmatrix} \cdot \operatorname{Im} \Psi^\theta(0, 0, 0) &= \begin{pmatrix} \xi \\ \tau \end{pmatrix} \cdot \begin{pmatrix} \operatorname{Im} \Psi(0, 0) \cos \theta - \operatorname{Re} \Psi(0, 0) \sin \theta \\ \sin \theta \end{pmatrix} \\ &= \xi \cdot \operatorname{Im} \Psi(0, 0) \cos \theta + (\tau - \xi \cdot \operatorname{Re} \Psi(0, 0)) \sin \theta. \end{aligned}$$

So the condition

$$\begin{pmatrix} \xi \\ \tau \end{pmatrix} \cdot \operatorname{Im} \Psi^\theta(0, 0, 0) \neq 0$$

for some $\theta \in [0, 2\pi)$ is equivalent to saying that $\sigma(0, 0, \xi, \tau) \neq 0$, i.e., $(0, 0, \xi, \tau)$ is not in the characteristic set of L .

Now, considering that a C^1 function $h(x, t)$ such that Lh is t -flat at $t = 0$ does not necessarily have the property that $L^\theta h$ is s -flat at $s = 0$, we make the following assumption: *there exists a C^1 function $h^\theta(x, t, s)$ such that $h^\theta(x, t, 0) = h(x, t)$ and $L^\theta h^\theta$ is s -flat at $s = 0$.*

To conclude, if $(0, 0, \xi^0, \tau^0)$ is not in the characteristic set of L , then there is $\theta \in [0, 2\pi)$ such that

$$\begin{pmatrix} \xi^0 \\ \tau^0 \end{pmatrix} \cdot \operatorname{Im} \Psi^\theta(0, 0, 0) \neq 0.$$

Without loss of generality we may assume that

$$\begin{pmatrix} \xi^0 \\ \tau^0 \end{pmatrix} \cdot \operatorname{Im} \Psi^\theta(0, 0, 0) < 0$$

and we can apply Lemma 2.1. There exist a conic neighborhood Γ of (ξ^0, τ^0) in $\mathbb{R}^{m+1} \setminus \{0\}$ and a neighborhood $\mathcal{O}$ of the origin in $\mathbb{C}^{m+1}$ such that

$$\begin{aligned} \mathcal{F} h^\theta(0; z, w, \xi, \tau) \\ &= \int_{B \times J} e^{i[\xi \cdot (z-x) + \tau(w-t)] - |(\xi, \tau)|[(z-x)^2 + (w-t)^2]} \underbrace{h^\theta(x, t, 0)}_{=h(x, t)} dx dt \\ &= \mathcal{F} h(z, w, \xi, \tau) \quad (\text{complete FBI transform}) \end{aligned}$$

is rapidly decreasing in $(\xi, \tau) \in \Gamma$ for all $(z, w) \in \mathcal{O}$. Using the inversion formula of the FBI transform we may write (cf. [3], Theorem 6.5, page 42) $h = h_0 + h_1 + \cdots + h_n$ in a neighborhood of the origin, where $h_0 \in C^\infty$ and $h_1, \dots, h_n$ are distribution boundary values of holomorphic functions defined in wedges such that $(0, 0, \xi^0, \tau^0) \notin WF_A(h_j)$, for all $j = 1, \dots, n$. The C^∞ part h_0 comes from the rapid decay of the FBI transform of h for $(\xi, \tau) \in \Gamma$ and $(z, w) \in \mathcal{O}$ (as opposed to exponential decay, in which case we would get $h_0 \in C^\omega$). Since $WF(v) \subset WF_A(v)$ for any distribution v , we obtain $(0, 0, \xi^0, \tau^0) \notin WF(h)$.

We summarize all this in the following lemma.

Lemma 2.2. *For each $\theta \in [0, 2\pi)$ define $L^\theta = \frac{\partial}{\partial s} - e^{-i\theta} L$ and suppose there exist $\Psi_1^\theta, \dots, \Psi_{m+1}^\theta \in C^1(\Omega \times J)$ such that $Z^\theta = (x, t) + s\Psi^\theta(x, t, s)$ is an approximate solution of $L^\theta Z^\theta = 0$ in the sense that $L^\theta Z^\theta$ is s -flat at $s = 0$. Suppose moreover that there exists $h^\theta \in C^1(\Omega \times J)$ such that $h^\theta(x, t, 0) =$*

$h(x, t)$ and $L^\theta h^\theta$ is s -flat at $s = 0$. Then

$$WF(h)|_0 \subset (\text{char } L)|_0.$$

3. EXISTENCE OF APPROXIMATE SOLUTIONS

Consider the complex vector field

$$L = \frac{\partial}{\partial t} + \sum_{j=1}^N a_j(x, t, \zeta) \frac{\partial}{\partial x_j} + \sum_{k=1}^M b_j(x, t, \zeta) \frac{\partial}{\partial \zeta_k}$$

where the coefficients a_j and b_k are C^∞ in the variables $(x, t) \in \Omega \times J \subset \mathbb{R}^N \times \mathbb{R}$ and holomorphic in the variable $\zeta \in \mathcal{N} \subset \mathbb{C}^M$.

Let $f(x, \zeta)$ be a C^∞ function defined on $\Omega \times \mathcal{N}$, holomorphic in ζ , and define recursively $u_0(x, \zeta) = f(x, \zeta)$ and

$$\begin{aligned} u_j(x, \zeta) = & -\frac{1}{j} \sum_{p+q=j-1} \frac{1}{q!} \left[\sum_{k=1}^M \frac{\partial u_p}{\partial x_k}(x, \zeta) \frac{\partial^q a_k}{\partial t^q}(x, 0, \zeta) \right. \\ & \left. + \sum_{k=1}^N \frac{\partial u_p}{\partial \zeta_k}(x, \zeta) \frac{\partial^q b_k}{\partial t^q}(x, 0, \zeta) \right], \quad j \geq 1. \end{aligned}$$

Note that the functions $u_j(x, \zeta)$ are C^∞ and holomorphic in ζ . Let $\chi \in C_c^\infty(\mathbb{R})$ be such that $\chi \geq 0$, $\chi \equiv 1$ in $[-\frac{1}{2}, \frac{1}{2}]$ and $\text{supp } \chi \subset [-1, 1]$. Then there exists a sequence $R_j > 1$, $R_j \nearrow +\infty$, such that the series below is convergent in C^∞ :

$$u(x, t, \zeta) = \sum_{j=0}^{\infty} \chi(R_j t) u_j(x, \zeta) t^j.$$

Since each u_j is holomorphic in ζ , it follows that u is C^∞ and holomorphic in ζ . Moreover, by the way the functions u_j are defined, u is an approximate solution of $Lu = 0$ in the sense of (1) with the property that $u(x, 0, \zeta) = f(x, \zeta)$. This construction will be used in the following section.

4. APPLICATION

In this section we will follow very closely §2 of Hanges and Treves [2]. Let $\Omega \subset \mathbb{R}^{m+1}$ be a neighborhood of the origin and suppose $u \in C^2(\Omega)$ is a solution of

$$(5) \quad u_t = f(x, t, u, u_x)$$

where $f(x, t, \zeta_0, \zeta)$ is a C^∞ function in the variables $(x, t) \in \Omega$ and holomorphic in the variables

$$(\zeta_0, \zeta) \in \mathcal{N} \subset \mathbb{C} \times \mathbb{C}^m, \quad (a, w) = (u(0, 0), u_x(0, 0)) \in \mathcal{N}.$$

Consider

$$(6) \quad \mathcal{L} = \frac{\partial}{\partial t} - \sum_{j=1}^m \left(\frac{\partial f}{\partial \zeta_j} \right) (x, t, \zeta_0, \zeta) \frac{\partial}{\partial x_j}$$

and

$$L^u = \frac{\partial}{\partial t} - \sum_{j=1}^m \left(\frac{\partial f}{\partial \zeta_j} \right) (x, t, u, u_x) \frac{\partial}{\partial x_j}.$$

Let $v = (u, u_x)$. It follows (cf. [2]) that

$$(7) \quad L^u v = g(x, t, v)$$

where

$$g_0(x, t, \zeta_0, \zeta) = f(x, t, \zeta_0, \zeta) - \sum_{j=1}^m \zeta_j \frac{\partial f}{\partial \zeta_j} (x, t, \zeta_0, \zeta),$$

$$g_i(x, t, \zeta_0, \zeta) = f_{x_i}(x, t, \zeta_0, \zeta) - \zeta_i \frac{\partial f}{\partial \zeta_0} (x, t, \zeta_0, \zeta), \quad i = 1, \dots, m.$$

Consider now the principal part of the holomorphic *Hamiltonian* of (7)

$$H = \mathcal{L} + g_0 \frac{\partial}{\partial \zeta_0} + \sum_{j=1}^m g_j \frac{\partial}{\partial \zeta_j}.$$

We have that (7) implies (cf. [2] equation (2.5))

$$\mathcal{L}^v \Psi^v = (H\Psi)^v$$

where $\Psi(x, t, \zeta_0, \zeta)$ is any C^∞ function in $(x, t) \in \Omega$ and holomorphic in $(\zeta_0, \zeta) \in \mathcal{N}$, with the notation $\Psi^v(x, t) = \Psi(x, t, v(x, t))$. Here $\mathcal{L}^v$ denotes the vector field L^u .

Let $Z_j(x, t, \zeta_0, \zeta)$, $j = 1, \dots, m$, and $\Xi_k(x, t, \zeta_0, \zeta)$, $k = 0, \dots, m$, be approximate solutions of $H\Psi = 0$ in the sense of (1) such that $Z_j(x, 0, \zeta_0, \zeta) = x_j$, $j = 1, \dots, m$, and $\Xi_k(x, 0, \zeta_0, \zeta) = \zeta_k$, $k = 0, \dots, m$.

Let $\tilde{Z}(z, t, \zeta_0, \zeta)$ and $\tilde{\Xi}(z, t, \zeta_0, \zeta)$ ($z = x + iy \in \mathbb{R}^m \oplus i\mathbb{R}^m$) be almost complex extensions of $Z(x, t, \zeta_0, \zeta)$ and $\Xi(x, t, \zeta_0, \zeta)$ respectively, i.e., $\tilde{Z}(x, t, \zeta_0, \zeta) = Z(x, t, \zeta_0, \zeta)$, $\tilde{\Xi}(x, t, \zeta_0, \zeta) = \Xi(x, t, \zeta_0, \zeta)$, and for all $\nu \in \mathbb{N}$ there exists $C_\nu > 0$ such that for $j = 1, \dots, m$ we have

$$\left| \frac{\partial}{\partial \bar{z}_j} \tilde{Z}(z, t, \zeta_0, \zeta) \right| \leq C_\nu |\operatorname{Im} z_j|^\nu$$

and

$$\left| \frac{\partial}{\partial \bar{z}_j} \tilde{\Xi}(z, t, \zeta_0, \zeta) \right| \leq C_\nu |\operatorname{Im} z_j|^\nu.$$

Since

$$\frac{\partial(\tilde{Z}, \tilde{\Xi})}{\partial(z, \zeta_0, \zeta)}$$

is non-singular near $t = 0$, we may solve

$$\begin{cases} \tilde{Z}(z, t, \zeta_0, \zeta) = \tilde{Z}, \\ \tilde{\Xi}(z, t, \zeta_0, \zeta) = \tilde{\Xi} \end{cases}$$

with respect to (z, ζ_0, ζ) in a neighborhood of $(0, a, w)$ by the implicit function theorem and get

$$\begin{cases} z = P(\tilde{Z}, t, \tilde{\Xi}), \\ (\zeta_0, \zeta) = Q(\tilde{Z}, t, \tilde{\Xi}), \end{cases}$$

with $P(0, 0, \zeta_0, \zeta) = 0$ and $Q(0, 0, a, w) = (a, w)$. Substituting in the original equations, we have

$$\begin{cases} \tilde{Z}(P(\tilde{Z}, t, \tilde{\Xi}), t, Q(\tilde{Z}, t, \tilde{\Xi})) = \tilde{Z}, \\ \tilde{\Xi}(P(\tilde{Z}, t, \tilde{\Xi}), t, Q(\tilde{Z}, t, \tilde{\Xi})) = \tilde{\Xi}, \end{cases}$$

and differentiating with respect to $\tilde{Z}$ we obtain

$$\begin{aligned} & \frac{\partial(\tilde{Z}, \tilde{\Xi})}{\partial(z, \zeta_0, \zeta)}(P(\tilde{Z}, t, \tilde{\Xi}), t, Q(\tilde{Z}, t, \tilde{\Xi})) \frac{\partial(P, Q)}{\partial \tilde{Z}}(\tilde{Z}, t, \tilde{\Xi}) \\ & + \frac{\partial(\tilde{Z}, \tilde{\Xi})}{\partial(\bar{z}, \bar{\zeta}_0, \bar{\zeta})}(P(\tilde{Z}, t, \tilde{\Xi}), t, Q(\tilde{Z}, t, \tilde{\Xi})) \frac{\partial(P, \bar{Q})}{\partial \bar{\tilde{Z}}}(\tilde{Z}, t, \tilde{\Xi}) = 0. \end{aligned}$$

If $A(z, t, \zeta_0, \zeta)$ denotes a generic entry of the matrix

$$\frac{\partial(\tilde{Z}, \tilde{\Xi})}{\partial(\bar{z}, \bar{\zeta}_0, \bar{\zeta})}(z, t, \zeta_0, \zeta),$$

then $|A(z, t, \zeta_0, \zeta)| \leq C_\nu |\operatorname{Im} z|^\nu$. It follows, in particular, that

$$\left| \frac{\partial Q_0}{\partial \tilde{Z}_j}(\tilde{Z}, t, \tilde{\Xi}) \right| \leq C'_\nu |\operatorname{Im} P(\tilde{Z}, t, \tilde{\Xi})|^\nu \quad \forall j = 1, \dots, m.$$

Now consider

$$\Psi(z, t, \zeta_0, \zeta) = Q_0(\tilde{Z}(z, t, \zeta_0, \zeta), 0, \tilde{\Xi}(z, t, \zeta_0, \zeta))$$

and observe that

$$\begin{aligned} \Psi^v(x, 0) &= \Psi(x, 0, u(x, 0), u_x(x, 0)) \\ &= Q_0(\tilde{Z}(x, 0, u(x, 0), u_x(x, 0)), 0, \tilde{\Xi}(x, 0, u(x, 0), u_x(x, 0))) \\ &= u(x, 0). \end{aligned}$$

Note that H has no derivation on $y = \operatorname{Im} z$, so that

$$H[\Psi(x, t, \zeta_0, \zeta)] = (H\Psi)(x, t, \zeta_0, \zeta).$$

This implies that $H\tilde{Z}(x, t, \zeta_0, \zeta)$ and $H\tilde{\Xi}(x, t, \zeta_0, \zeta)$ are t -flat at $t = 0$. We have

$$H\Psi = \sum_{j=1}^m \left(\frac{\partial Q_0}{\partial \tilde{Z}_j} H\tilde{Z}_j + \frac{\partial Q_0}{\partial \bar{\tilde{Z}}_j} H\bar{\tilde{Z}}_j \right) + \sum_{k=0}^m \frac{\partial Q_0}{\partial \tilde{\zeta}_k} H\tilde{\Xi}_k.$$

Also,

$$\begin{aligned} P(x, 0, \zeta_0, \zeta) &= P(Z(x, 0, \zeta_0, \zeta), 0, \Xi(x, 0, \zeta_0, \zeta)) \\ &= P(\tilde{Z}(x, 0, \zeta_0, \zeta), 0, \tilde{\Xi}(x, 0, \zeta_0, \zeta)) = x, \end{aligned}$$

so, by the mean value inequality,

$$|\operatorname{Im} P(\tilde{Z}, 0, \tilde{\Xi})| \leq \operatorname{const} |\operatorname{Im} \tilde{Z}|.$$

Since

$$|\operatorname{Im} \tilde{Z}(x, t, \zeta_0, \zeta)| = |\operatorname{Im} Z(x, t, \zeta_0, \zeta)| \leq \operatorname{const} |t|,$$

we have that

$$\frac{\partial Q_0}{\partial \tilde{Z}_j}(\tilde{Z}(z, t, \zeta_0, \zeta), 0, \tilde{\Xi}(z, t, \zeta_0, \zeta))$$

is t -flat at $t = 0$, which implies that for all $\nu \in \mathbb{N}$ there exists $C''_\nu > 0$ such that

$$|(H\Psi)(x, t, \zeta_0, \zeta)| \leq C''_\nu |t|^\nu.$$

Hence $L^u \Psi^v = \mathcal{L}^v \Psi^v = (H\Psi)^v$ is t -flat at $t = 0$, and so we have found $h(x, t)$ (given by $\Psi^v(x, t)$) such that $L^u h$ is t -flat at $t = 0$ and $h(x, 0) = u(x, 0)$.

Now $u(x, t)$ is also a solution of the equation

$$u_s = e^{-i\theta}(u_t - f(x, t, u, u_x)),$$

which is of the same kind as (5), and the associated vector field $\mathcal{L}^\theta$ as in (6) is given by

$$\mathcal{L}^\theta = \frac{\partial}{\partial s} - e^{-i\theta} \mathcal{L},$$

with $\mathcal{L}$ as before. Note that

$$(\mathcal{L}^\theta)^v = \frac{\partial}{\partial s} - e^{-i\theta} \mathcal{L}^v = \frac{\partial}{\partial s} - e^{-i\theta} L^u = (L^u)^\theta.$$

It follows that there exists a C^1 function $h^\theta(x, t, s)$ such that

$$(L^u)^\theta h^\theta = \left(\frac{\partial}{\partial s} - e^{-i\theta} L^u \right) h^\theta$$

is s -flat at $s = 0$ and $h^\theta(x, t, 0) = u(x, t)$. We apply Lemma 2.2 and conclude that $WF(u)|_0 \subset \operatorname{char} L^u|_0$.

By translation we may apply the same argument for all points of Ω and state

Theorem 4.1. *Let $u \in C^2(\Omega)$ be a solution of (5). Then the C^∞ wave-front set of u is contained in the characteristic set of the linearized operator L^u .*

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