ON THE INEQUALITY $|p(z)| \leq p(|z|)$ FOR POLYNOMIALS

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ABSTRACT. We derive a simple test to check if the inequality $|p(z)| \leq p(|z|)$ holds for all sufficiently small or sufficiently large values of $|z|$, where p is a Laurent polynomial in one variable with real coefficients and $z \in \mathbb{C}$. Some examples and applications are also given.

1. INTRODUCTION

Let $p \in \mathbb{R}[x^\pm]$, i.e., p is a Laurent polynomial in one variable with real coefficients. We say that p is *numerically positive* if $p(r) > 0$ for $r > 0$. A much stronger notion of positivity for polynomials is obtained by requiring that all coefficients be positive. Polynomials with positive coefficients appear in several branches of mathematics and have been studied by many authors [4, 5, 9, 11]. In [2], we discussed some intermediate notions of positivity, two of which are

$$w(*) \quad |p(z)| \leq p(|z|) \quad \text{for all } z \in \mathbb{C} \setminus \{0\}$$

and

$$(*) \quad |p(z)| < p(|z|) \quad \text{for all } z \in \mathbb{C} \setminus \mathbb{R}^+.$$

Condition $w(*)$ and $(*)$ above arise naturally in more than one context. If A is a square matrix whose entries are polynomials with positive, integral coefficients, v is a vector of numerically positive polynomials, and $p \in \mathbb{R}[x^\pm]$ satisfies $Av = pv$, then p is the *beta function* of the matrix A , an important invariant of the Markov chain determined by A (see [3, 8, 10]). It is easy to see from Perrom-Frobenius theory that any such polynomial satisfies $w(*)$, and it is shown in [3] that if in addition the additive group generated by all differences of exponents appearing in p is $\mathbb{Z}$, then p satisfies $(*)$. However, an example in this paper will show that not all polynomials satisfying $(*)$ are beta functions.

In [2], we show that for numerically positive polynomials condition $w(*)$ is equivalent to the following growth rate condition on the coefficients of large

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powers of p :

for all $r > 0$, $\lim_{n \rightarrow \infty} \frac{1}{n} \log \left(\max_k r^k |(p^n)_k| \right)$ exists and equals $\log p(r)$,

where $(p^n)_k$ is the coefficient of x^k in p^n . Moreover, a local version of $w(*)$ (restricting the variable to be in a neighborhood of the positive real axis) characterizes those polynomials p for which the Legendre transformation $\Lambda_p(x) = xp'(x)/p(x)$ has strictly positive derivative on $(0, \infty)$. These results (more precisely, their appropriate generalizations) were in fact proved for polynomials in several variables.

In [11], Odlyzko and Richmond derived results on strong unimodality for high convolutions of finite probability distributions, by studying the asymptotic behavior of the coefficients of large powers of a polynomial in one variable with positive coefficients. Similar estimates for the coefficients can be derived for polynomials satisfying $(*)$. We will discuss this and other results in a subsequent paper.

In view of the results mentioned above, it is of interest to decide when a given polynomial in one variable satisfies conditions $w(*)$ or $(*)$. In general, this can clearly be a difficult problem. We refer to [2] for some discussion, and also for a class of polynomials for which $w(*)$ and $(*)$ can be easily checked.

The main goal of this paper is to derive a criterion that allows us to readily check if $w(*)$ or $(*)$ hold for all small enough or large enough values of $|z|$. Informally, the criterion consists in identifying certain “crucial” exponents, using a simple divisibility test, and then checking the sign of the corresponding coefficients. We will describe it more precisely in the next section, after we fix the necessary notation. In section 3 we will prove the criterion, and in section 4 we will discuss some examples.

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2. DEFINITIONS AND STATEMENT OF THE CRITERION

We denote by $\mathbb{R}^+[x^\pm]$ the subsemiring of $\mathbb{R}[x^\pm]$ consisting of polynomials with nonnegative coefficients. If $p \in \mathbb{R}[x^\pm]$ and $k \in \mathbb{Z}$, we write p_k for the coefficient of x^k in p , and we put $\text{Log}(p) = \{k \in \mathbb{Z} : p_k \neq 0\}$, $m = m(p) = \min \text{Log}(p)$, $M = M(p) = \max \text{Log}(p)$. So if p is not identically zero we have $p(x) = \sum_{k=m}^M p_k x^k$, with $p_m \neq 0 \neq p_M$. Multiplication by a monomial does not affect whether or not p satisfies $w(*)$ or $(*)$, so, multiplying p by x^{-m} , we may assume that $m = 0$, i.e., $p \in \mathbb{R}[x]$ and $p(0) \neq 0$.

If $\text{Log}(p)$ is the single point $\{0\}$, then p is a constant. So to avoid trivialities we will assume that $\text{Log}(p)$ has at least two points. If $m = 0$ and $\gcd(\text{Log}(p) \setminus \{0\}) = d > 1$, then we can write $p(x) = q(x^d)$ for some $q \in \mathbb{R}[x]$ such that $\gcd(\text{Log}(q) \setminus \{0\}) = 1$. Clearly p satisfies $w(*)$ if and only if q does, while p always trivially fails to satisfy $(*)$ (at the points $re^{2\pi i k/d}$, $1 \leq k \leq d-1$) . In order to consider only non-trivial violations of $(*)$, we therefore also assume that $\gcd(\text{Log}(p) \setminus \{0\}) = 1$. While this last condition clearly implies $(*)$ when all coefficients are positive, we remark that in general it and $w(*)$ together do not imply $(*)$. An example is $2 + 4x - 3x^2 + 4x^3 + 2x^4$, discussed in [2].

It will be our standing assumption in this and the next section that p is

a non-constant element of $\mathbb{R}[x]$ such that $p_0 = p(0) \neq 0$, and p is not a polynomial in x^d for any $d > 1$.

Let $l = l(p) = \min\{k \in \text{Log}(p) : k > 0\}$. If $l > 1$, it is easy to check (as a consequence of the fact that $\gcd(\text{Log}(p) \setminus \{0\}) = 1$) that for $1 \leq j \leq [l/2]$ (where $[l/2]$ is the greatest integer not exceeding $l/2$), not all elements of $\{jk : k \in \text{Log}(p) \setminus \{0\}\}$ are multiples of l . Put $l_j = \min\{k \in \text{Log}(p) \setminus \{0\} : jk/l \notin \mathbb{N}\}$, for $1 \leq j \leq [l/2]$, and $l_0 = l$. Note that if $p_0 < 0$, then p is not numerically positive, and in particular it does not satisfy $w(*)$. Hence we need only consider the case $p_0 > 0$. We now state our criterion, with all assumptions spelled out.

Theorem 1. *If $p \in \mathbb{R}[x]$ is not a constant, $p_0 > 0$, and $\gcd(\text{Log}(p) \setminus \{0\}) = 1$, then the following are equivalent:*

- (i) $|p(re^{i\theta})| \leq p(r)$ for all θ and all small enough $r > 0$.
- (ii) $|p(re^{i\theta})| < p(r)$ for all $\theta \in (0, 2\pi)$ and all small enough $r > 0$.
- (iii) $p_{l_j} > 0$ for $0 \leq j \leq [l/2]$.

Now define $\tilde{p}(x) = x^M p(1/x)$. So $\tilde{p}$ is p with the coefficients read in reverse order, and it also satisfies our standing assumptions. Clearly $|p(re^{i\theta})| \leq p(r)$ for large $r > 0$ if and only if $|\tilde{p}(re^{i\theta})| \leq \tilde{p}(r)$ for small $r > 0$. So Theorem 1 applied to $\tilde{p}$ can be used to check if $w(*)$ or $(*)$ hold for all large values of $|z|$.

3. PROOF OF THEOREM 1

We begin by studying the function $\theta \mapsto |p(re^{i\theta})|^2$ for small $r > 0$. Since $|p(re^{-i\theta})| = |p(re^{i\theta})|$, it is enough to consider $\theta \in [0, \pi]$. Given a function f defined on a neighborhood of $r = 0$, we will use the notation $O(f(r))$ to denote any function (of r and θ) that is bounded (uniformly in $\theta \in [0, \pi]$) by a constant multiple of $|f(r)|$ as $r \rightarrow 0^+$.

Put $g_r(\theta) = |p(re^{i\theta})|^2$. Then

$$g_r(\theta) = \sum_{j,k} p_j p_k r^{j+k} \cos((j-k)\theta) = p_0^2 + 2p_0 p_l r^l \cos(l\theta) + O(r^{l+1}),$$

and so $g_r(0) - g_r(\theta) = 2p_0 p_l r^l (1 - \cos(l\theta)) + O(r^{l+1})$. We have already remarked that $p_0 > 0$ is necessary for p to be numerically positive, and so we conclude that both $p_0 > 0$ and $p_l > 0$ are necessary for p to satisfy $w(*)$ at small r .

Proposition 2. *Suppose $p_0 > 0$, $p_l > 0$.*

(i) *Suppose $l = 1$. Then there is some $r_0 > 0$ such that the function $\theta \mapsto g_r(\theta)$ is strictly decreasing on $[0, \pi]$ for all $r \in (0, r_0]$.*

(ii) *Suppose $l > 1$. Then there is some $r_0 > 0$ such that for each $r \in (0, r_0]$, there are precisely $[l/2]$ local maxima of the function $\theta \mapsto g_r(\theta)$ in $(0, \pi]$. Denote them by $\tilde{\theta}_j(r)$, $1 \leq j \leq [l/2]$. Then the functions $r \mapsto \tilde{\theta}_j(r)$ are continuous on $(0, r_0]$, and*

$$\lim_{r \rightarrow 0^+} \tilde{\theta}_j(r) = \frac{2\pi j}{l}, \quad 1 \leq j \leq [l/2].$$

Moreover, if l is even, then $\tilde{\theta}_{l/2}(r) = \pi$ for all $r \in (0, r_0]$.

Proof. From the expression above for $g_r(\theta)$, we find

$$g'_r(\theta) = -2p_0 p_l l r^l \sin(l\theta) + O(r^{l+1})$$

and

$$g_r''(\theta) = -2p_0p_l l^2 r^l \cos(l\theta) + O(r^{l+1}).$$

So we can write

$$(1) \quad -\frac{1}{2p_0p_l l r^l} g_r'(\theta) = \sin(l\theta) + rG(r, \theta),$$

$$(2) \quad -\frac{1}{2p_0p_l l^2 r^l} g_r''(\theta) = \cos(l\theta) + rH(r, \theta),$$

for some functions G, H that are polynomials in r and trigonometric polynomials in θ .

Suppose first that $l = 1$. Using the identity

$$\frac{\cos(k\theta_1) - \cos(k\theta_2)}{\cos(\theta_1) - \cos(\theta_2)} = U_k\left(\frac{\theta_1 + \theta_2}{2}\right) U_k\left(\frac{\theta_1 - \theta_2}{2}\right),$$

where $U_k(\theta) = \sin(k\theta)/\sin\theta$, and the fact that $|U_k(\theta)| \leq k$, the expression for $g_r(\theta)$ gives us

$$g_r(\theta_1) - g_r(\theta_2) = 2p_0p_l r(\cos(\theta_1) - \cos(\theta_2))(1 + O(r)).$$

So the statement in (i) follows. Now suppose $l > 1$. Let $\theta_j = 2\pi j/l$, $1 \leq j \leq [l/2]$. Consider the function

$$h(r, \theta) = \sin(l\theta) + rG(r, \theta).$$

Then

$$h(0, \theta_j) = 0, \quad \frac{\partial h}{\partial \theta}(0, \theta_j) = l > 0, \quad \text{for } 1 \leq j \leq [l/2].$$

By the implicit function theorem, for each j , we can find a neighborhood V_j of 0, a neighborhood U_j of θ_j , and a continuous function $\tilde{\theta}_j: V_j \rightarrow U_j$ such that $\tilde{\theta}_j(0) = \theta_j$, and $h(r, \tilde{\theta}_j(r)) = 0$ for all $r \in V_j$. Moreover, $\tilde{\theta}_j$ is uniquely determined by the conditions $\tilde{\theta}_j(V_j) \subset U_j$ and $h(r, \tilde{\theta}_j(r)) = 0$ for all $r \in V_j$.

Since $\cos(l\tilde{\theta}_j(r)) \rightarrow 1$ as $r \rightarrow 0^+$, we have $\frac{\partial h}{\partial \theta}(r, \tilde{\theta}_j(r)) > 0$ for all small enough $r > 0$. Using (1) and (2), we can find some $r_0 > 0$ such that for each j , $\tilde{\theta}_j$ is defined on $[0, r_0]$, and $g_r'(\tilde{\theta}_j(r)) = 0$, $g_r''(\tilde{\theta}_j(r)) < 0$ for all $r \in (0, r_0]$, i.e., $\tilde{\theta}_j(r)$ is a local maximum of g_r , and from the uniqueness property of $\tilde{\theta}_j$ mentioned above, it follows that there are no other local maxima of g_r in $(0, \pi]$.

Since $g_r'(\pi) = 0$ for all $r \geq 0$, and $g_r''(\pi)$ has the opposite sign of $\cos(l\pi)$ for small r , it follows from the above that when l is even, $\tilde{\theta}_{l/2}(r) = \pi$ for small $r > 0$. $\square$

According to the previous proposition, if $l = 1$, then p satisfies $w(*)$ (or $(*)$) for all small $r > 0$ if and only if $p_1 > 0$. So we now assume $l > 1$. In order to check $w(*)$ for small r , it is enough to check it at the points $re^{i\tilde{\theta}_j(r)}$, $1 \leq j \leq [l/2]$. To do this, we will need to know the rate of convergence of $\tilde{\theta}_j(r)$ to $\theta_j = 2\pi j/l$ as $r \rightarrow 0^+$. This is done in the next lemma.

If $l = 2$, then the only local maximum in $(0, \pi]$ is at π , and $\tilde{\theta}_1(r) = \pi$ for all small r . If $l > 2$, let

$$l'_j = \min\{k \in \text{Log}(p) \setminus \{0\} : 2jk/l \notin \mathbb{N}\},$$

for $1 \leq j < l/2$ (the existence of l'_j follows from the fact that $\gcd(\text{Log}(p) \setminus \{0\}) = 1$). Clearly l'_j is always at least as large as l_j previously defined, and $l'_j = l_j$ when l is odd. Now put $\varepsilon_j(r) = \tilde{\theta}_j(r) - \theta_j$. The next lemma says that $\varepsilon_j(r)$ is of order $r^{l'_j-l}$ for small $r > 0$.

Lemma 3. *If $l > 2$ and $1 \leq j < l/2$, then*

$$\tilde{\theta}_j(r) = \theta_j - r^{l'_j-l} \frac{l'_j p_{l'_j}}{p_l l^2} \sin(\theta_j l'_j) + O(r^{l'_j-l+1}).$$

Proof. Fix j , $1 \leq j < l/2$. Recall that for small $r > 0$, $\tilde{\theta}_j(r)$ is a local maximum of the function $\theta \mapsto |p(re^{i\theta})|^2 = g_r(\theta)$. If $1 \leq k < l'_j$, then $\sin(k\theta_j) = 0$, and it follows that

$$\sin(k\tilde{\theta}_j(r)) = \cos(k\theta_j)(k\varepsilon_j(r) + O(\varepsilon_j^3(r))).$$

Using this in the equation that we find by setting to zero the derivative of $g_r(\theta) = (\sum_k p_k r^k \cos(k\theta))^2 + (\sum_k p_k r^k \sin(k\theta))^2$ evaluated at $\theta = \tilde{\theta}_j(r)$, we find, after a few lines of manipulations,

$$p_0 p_l \varepsilon_j(r) l^2 r^l (1 + O(r) + O(\varepsilon_j^2(r))) + l'_j p_0 p_{l'_j} r^{l'_j} \sin(l'_j \tilde{\theta}_j(r)) + O(r^{l'_j+1}) = 0,$$

and the result follows from this. $\square$

Theorem 1 is a direct consequence of the next proposition.

Proposition 4. *Suppose that $l > 1$. Then for each $j = 1, \dots, [l/2]$, we have*

$$p(r)^2 - |p(re^{i\tilde{\theta}_j(r)})|^2 = 2p_0 p_{l_j} r^{l_j} (1 - \cos(l_j \theta_j)) + O(r^{l_j+1})$$

as $r \rightarrow 0^+$.

Proof. Fix j such that $1 \leq j \leq [l/2]$. In the following, if l is even and $j = l/2$, we put $l'_j = +\infty$, and by convention $r^{+\infty} = 0$.

Note that $\cos(k\tilde{\theta}_j(r)) = \cos(k\varepsilon_j(r))$ and $\sin(k\tilde{\theta}_j(r)) = \sin(k\varepsilon_j(r))$ for $0 \leq k < l_j$, while $\cos(l_j \tilde{\theta}_j(r)) = \cos(l_j \theta_j) + O(r)$. If $l > 2$ and $1 \leq j < l/2$, then Lemma 3 gives us $\varepsilon_j(r) = O(r^{l'_j-l})$, while if l is even and $j = l/2$, then $\varepsilon_j(r) = 0$ for all small $r > 0$. So in every case $\varepsilon_j(r) = O(r^{l'_j-l})$.

We have

$$p(r)^2 - |p(re^{i\tilde{\theta}_j(r)})|^2 = 2 \sum_{n \geq 0} \sum_{k \geq n+1} p_k p_n r^{k+n} (1 - \cos((n-k)\tilde{\theta}_j(r))).$$

The term in the double sum corresponding to $n = 0$ and $k = l_j$ is

$$2p_0 p_{l_j} r^{l_j} (1 - \cos(l_j \tilde{\theta}_j(r))) = 2p_0 p_{l_j} r^{l_j} (1 - \cos(l_j \theta_j)) + O(r^{l_j+1}).$$

The remaining terms can be partitioned into three sets: those with $k < l_j$ and $n < l_j$, which are $O(r^{2l'_j-l})$ (because $1 - \cos((k-n)\tilde{\theta}_j(r)) = O(r^{2l'_j-2l})$); those with $n = 0$ and $k > l_j$, which are $O(r^{l'_j+1})$; and those with $n \geq 1$ and $k \geq l_j$, which are $O(r^{l'_j+1})$. Since $2l'_j - l \geq 2l_j - l \geq l_j + 1$, the result follows. $\square$

4. EXAMPLES AND APPLICATIONS

As a first application we will show that there are finite subsets L of $\mathbb{N}$ with arbitrarily large cardinality, but such that any polynomial p with $\text{Log}(p) = L$ satisfies $w(*)$ if and only if all its coefficients are positive.

Let $n \geq 1$ be given. Take distinct primes a, b with $a < b$. Let $k_i = a^{n-i}b^{i-1}$, $1 \leq i \leq n$, and $k_0 = 0$. Then $\gcd\{k_i: 1 \leq i \leq n\} = 1$. Suppose $p \in \mathbb{R}[x]$ and $\text{Log}(p) = \{k_0, k_1, \dots, k_n\}$. For each i , $2 \leq i \leq n$, there is some j such that $1 \leq j \leq [k_1/2]$ and $l_j = k_i$ (namely $j = a^{i-2}$). Hence, by Theorem 1, p satisfies $w(*)$ for all small values of $|z|$ if and only if $p \in \mathbb{R}^+[x]$. A specific example is $p(x) = p_0 + p_{16}x^{16} + p_{24}x^{24} + p_{36}x^{36} + p_{54}x^{54} + p_{81}x^{81}$.

We say that a polynomial $p \in \mathbb{R}[x^\pm]$ is *eventually positive* if $p^n \in \mathbb{R}^+[x^\pm]$ for all large enough n .¹ It is clear that $w(*)$ is necessary for p to be eventually positive. However, $w(*)$ is preserved when taking sums of polynomials, while the property of being eventually positive is not, so for example $1 + x^3(2 + 2x - x^2 + 2x^3 + 2x^4)$ satisfies $w(*)$ (the square of the polynomial in brackets has no negative coefficients), but in any of its powers the coefficient of x^5 is negative.

We will now use Theorem 1 to give an example of a polynomial that satisfies $(*)$ but cannot be written as a sum of eventually positive polynomials. This example depends on calculations done by computer (to show that $(*)$ is satisfied). First we need a lemma. Recall that for p such that $|\text{Log}(p)| \geq 2$ we define $m(p) = \min \text{Log}(p)$, $l(p) = \min\{k \in \text{Log}(p): k > m(p)\}$, $M(p) = \max \text{Log}(p)$. Now we also define $L(p) = \max\{k \in \text{Log}(p): k < M(p)\}$.

Lemma 5. Let $p \in \mathbb{R}[x^\pm]$, with $\text{Log}(p)$ not a single point. Suppose that $p = \sum_{q \in \mathcal{F}} q$, where $\mathcal{F}$ is a finite set of polynomials satisfying $w(*)$. Then for each $q \in \mathcal{F}$, we must have $q_k = 0$ for $k < m(p)$, $m(p) < k < l(p)$, $L(p) < k < M(p)$, $k > M(p)$.

Proof. By Theorem 1, the first two non-zero and the last two non-zero coefficients of a polynomial satisfying $w(*)$ must be positive. Suppose that there is some $q \in \mathcal{F}$ with $q_k \neq 0$ for some $k < m(p)$. Let $j = \min\{i < m(p): q_i \neq 0 \text{ for some } q \in \mathcal{F}\}$. Since $p_j = 0$, we must have $\hat{q}_j < 0$ for some $\hat{q} \in \mathcal{F}$. But then $\hat{q}_i = 0$ for all $i < j$, and $\hat{q}$ cannot satisfy $w(*)$. So $q_i = 0$ for all $q \in \mathcal{F}$ and $i < m(p)$. If there is some $q \in \mathcal{F}$ with $q_k \neq 0$ for some k such that $m(p) < k < l(p)$, let $j = \min\{i: m(p) < i < l(p), q_i \neq 0 \text{ for some } q \in \mathcal{F}\}$. Then there is some $\hat{q} \in \mathcal{F}$ with $\hat{q}_j < 0$, and, as already proved, $\hat{q}_i = 0$ for $i < m(p)$. Since also $\hat{q}_i = 0$ if $m(p) < i < l(p)$, $\hat{q}_j$ is either the first or the second non-zero coefficient of $\hat{q}$, and being negative, $\hat{q}$ cannot satisfy $w(*)$. So we conclude that $q_k = 0$ for all $q \in \mathcal{F}$ and $m(p) < k < l(p)$.

To prove the corresponding result for $L(p) < k < M(p)$ and $k > M(p)$, note that $\tilde{p}(x) = x^{M(p)+m(p)}p(1/x)$ satisfies $w(*)$ if and only if p does, and if $\hat{q}(x) = q(1/x)$, then $\hat{q}_k = q_{-k}$. So we can use the first part of the proof applied to $\tilde{p}$. $\square$

Example 6. Let $p(x) = 2 + 2x^5 + x^6 - x^7 + x^8 + 2x^9 + 2x^{14}$. Then p satisfies $(*)$, but is not a sum of eventually positive polynomials.

Proof. The proof that p satisfies $(*)$ relies on the fact that

$$q(x) = (p(x))^2 - 4 - 4x^{28}^{12}$$

has only positive coefficients. While such a fact is readily checked with a computer, I have not checked the calculations by hand, nor do I know of another

¹D. Handelman has shown that if $p(1) > 0$ and $p^n \in \mathbb{R}^+[x^\pm]$ for some n , then p is eventually positive [6].

way to prove it. Since q has positive coefficients and

$$\gcd\{j - k : j, k \in \text{Log}(q)\} = 1,$$

it satisfies $(*)$, and hence so does p .

We now prove that p is not a sum of eventually positive polynomials. Suppose that $p = \sum_{q \in \mathcal{F}} q$, where $\mathcal{F}$ is a finite set of eventually positive polynomials. So each $q \in \mathcal{F}$ satisfies $w(*)$, and from Lemma 5 we have that $q_k = 0$ for all $q \in \mathcal{F}$ and $k < 0$, $k > 14$, $k \in \{1, 2, 3, 4, 10, 11, 12, 13\}$. There must be some $q \in \mathcal{F}$ with $q_7 < 0$. If $q_0 \neq 0$, then $q_0 > 0$ because q is numerically positive, and then $(q^n)_7 = nq_0^{n-1}q_7 < 0$ for all $n \in \mathbb{N}$, contradicting the fact that p is eventually positive. Hence $q_0 = 0$. In a similar way, we find $q_{14} = 0$. So $q(x) = q_5x^5 + q_6x^6 + q_7x^7 + q_8x^8 + q_9x^9$, with $q_7 < 0$. Hence q_5, q_6, q_8, q_9 must be positive. If $q_5 > 2$, there must be some $\hat{q} \in \mathcal{F}$ with $\hat{q}_5 < 0$, which is impossible because $\hat{q}_1 = \hat{q}_2 = \hat{q}_3 = \hat{q}_4 = 0$ and $\hat{q}_k = 0$ for $k < 0$. So $q_5 \leq 2$, and similarly $q_9 \leq 2$. If $q_6 > 1$, then there must be some $\hat{q} \in \mathcal{F}$ with $\hat{q}_6 < 0$. Now $\hat{q}_0 \leq 0$ or $\hat{q}_5 \leq 0$ is impossible, so we have $\hat{q}_0 > 0$, $\hat{q}_5 > 0$, $\hat{q}_6 < 0$. But then, by Theorem 1, $\hat{q}$ does not satisfy $w(*)$. So we must have $q_6 \leq 1$, and similarly $q_8 \leq 1$. But then $x^5(2 + x - x^2 + x^3 + 2x^4) - q(x)$ has positive coefficients, hence $Q(x) = 2 + x - x^2 + x^3 + 2x^4$ satisfies $w(*)$. However, if $\cos \theta = -1/8$, we find

$$Q(1)^2 - |Q(e^{i\theta})|^2 = 4 \cos \theta (1 - \cos \theta) (5 + 4 \cos \theta) (1 + 4 \cos \theta) = -\frac{81}{64} < 0.$$

(Alternatively, see Theorem 8.5 of [2].)

So we get a contradiction, and we conclude that p is not a sum of eventually positive polynomials. $\square$

Recall from the introduction that if $p \in \mathbb{R}[x^\pm]$ is a beta function (i.e., the pointwise spectral radius of a matrix whose entries are polynomials with positive, integral coefficients) and the group generated by $\{j - k : j, k \in \text{Log}(p)\}$ is $\mathbb{Z}$, then p satisfies $(*)$. We will now use Theorem 1 to show that the converse is false.

If p is a polynomial beta function, then there is a matrix A with entries in $\mathbb{Z}^+[x^\pm]$ and a vector v of numerically positive polynomials such that $Av = pv$. Hence $(A^n - A^m)v = (p^n - p^m)v$ holds for all n, m , and so if we can find $n > m$ such that each entry of $A^n - A^m$ has only positive coefficients, then $p^n - p^m$ is also a polynomial beta function. It is proved in [1, Theorem 8.6] that when $\min \text{Log}(p) = 0$ and A is aperiodic (i.e., some power of A has no zero entries), then such n and m can always be found (see also [7, Proposition 2.9] for a closely related result). The condition that A be aperiodic is really no restriction, because it can always be arranged by replacing p with some power of it. So we conclude that a necessary condition for $p \in \mathbb{R}[x]$ with $p(0) \neq 0$ to be a beta function is that $p^n - p^m$ satisfy $w(*)$ for some n, m with $n > m$. We will show by example that this is not guaranteed by $(*)$ alone. First we recall the following result mentioned in the introduction, and proved in [2].

Theorem 7. *Let $p \in \mathbb{R}[x^\pm]$ be numerically positive, and let*

$$\Lambda_p(x) = xp'(x)/p(x).$$

Then Λ_p has strictly positive derivative on $(0, \infty)$ if and only if for each $r > 0$, there is a neighborhood V of r in $\mathbb{C}$ such that $|p(z)| \leq p(|z|)$ for all $z \in V$.

We will also need the following lemma.

Lemma 8. Let $p \in \mathbb{R}[x]$. Suppose that p satisfies $w(*)$ for all large and all small values of $|z|$, and Λ'_p is strictly positive on $(0, \infty)$. If $q \in \mathbb{R}[x]$ is any polynomial that satisfies $(*)$, then there is some $n \in \mathbb{N}$ such that $p + nq$ satisfies $(*)$.

Proof. Let $r_0, R > 0$ be such that $|p(z)| \leq p(|z|)$ whenever $|z| \leq r_0$ or $|z| \geq R$. Using Theorem 7 and compactness of $[r_0, R]$, we can find some $\theta_0 \in (0, \pi)$ such that $|p(re^{i\theta})| \leq p(r)$ for all $(r, \theta) \in [r_0, R] \times [0, \theta_0]$. Since q satisfies $(*)$, there is some $\varepsilon > 0$ such that

$$\frac{|q(re^{i\theta})|}{q(r)} \leq 1 - \varepsilon \quad \text{for all } (r, \theta) \in [r_0, R] \times [\theta_0, \pi].$$

Let

$$K = \max \left\{ \frac{|p(re^{i\theta})|}{q(r)} : (r, \theta) \in [r_0, R] \times [\theta_0, \pi] \right\},$$

and take n such that $n > K/\varepsilon$. If $(r, \theta) \in [r_0, R] \times [\theta_0, \pi]$, and $z = re^{i\theta}$, we then have

$$\frac{|p(z) + nq(z)|}{p(r) + nq(r)} = \frac{\left| \frac{p(z)}{nq(r)} + \frac{q(z)}{q(r)} \right|}{1 + \frac{p(r)}{nq(r)}} < \frac{|p(z)|}{nq(r)} + \frac{|q(z)|}{q(r)} \leq \frac{K}{n} + 1 - \varepsilon < 1.$$

On the other hand, if $(r, \theta) \in (0, r_0] \times (0, \pi] \cup [r_0, \infty) \times (0, \pi] \cup [r_0, R] \times (0, \theta_0]$, we have $|p(z) + nq(z)| \leq |p(z)| + n|q(z)| < p(|z|) + nq(|z|)$, because q satisfies $(*)$. So the result follows. $\square$

Example 9. Let $p(x) = 1 + 10x^5 + 10x^7 - x^8 + 10x^9 + 10x^{10}$. Then there is some $k \in \mathbb{N}$ such that $q(x) = p(x) + kx^9(1+x)$ satisfies $(*)$, but $q^n - q^m$ does not satisfy $w(*)$ for any distinct, positive integers n, m . In particular, q is not a beta function.

Remark. Computer generated graphs convince me that p itself already satisfies $(*)$, but I cannot prove it.

Proof. We have $l(p) = 5$, $l_1(p) = l_2(p) = 7$, and $\tilde{p}(x) = x^{10}p(1/x)$ has $l(\tilde{p}) = 1$. By Theorem 1 and the remark following it, p satisfies $w(*)$ for all small enough and large enough values of $|z|$. Also, a long and tedious computation shows that

$$\begin{aligned} \frac{p(x)^2}{2x^4} \Lambda'_p(x) &= 125 + 213x^2 + 373x^4 + 500x^5 + 155x^7 \\ &\quad + 755x^9 + 1245x^{10} + 200x^{11} + 425x^{12} + 30x^{14} \\ &\quad + x^2(32 + 45x^5 + 20x^{10})(1 - x + x^2). \end{aligned}$$

So $\Lambda'_p(r) > 0$ for $r > 0$, and since $x^9(1+x)$ satisfies $(*)$, by Lemma 8 $q(x) = p(x) + kx^9(1+x)$ satisfies $(*)$ for some $k \in \mathbb{N}$. However, if $n > m$, we have $q^n(x) - q^m(x) = x^5(c_1 + c_2x^2 - c_3x^3 + x^4Q(x))$, where c_1, c_2, c_3 are positive and $Q \in \mathbb{R}[x]$. By Theorem 1, $w(*)$ fails for $q^n - q^m$ when $|z|$ is small. $\square$

Note that the previous example relies on the fact that $p(0) = 1$. While multiplication by a constant obviously does not affect $w(*)$ or $(*)$, the result

mentioned above (before Theorem 7) according to which $A^n - A^m$ has entries with positive coefficients for some $n > m$ fails (badly) for matrices of polynomials with rational coefficients. So for example the method above cannot be used to rule out $2 + 10x^5 + 10x^7 - x^8 + 10x^9 + 10x^{10}$ as a beta function. I don't know if this is a beta function.

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