THE STABILITY OF POSITIVE SEMIGROUPS ON L_p SPACES

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ABSTRACT. For a positive semigroup T_t on $L_p(\Omega, m)$ with generator A , the growth bound of (T_t) equals the spectral bound of A . In particular, if $s(A) < 0$, the mild solutions of the Cauchy problem $u' = Au$ are asymptotically stable.

1. INTRODUCTION

Let T_t be a c_0 -semigroup on a Banach space X with generator A . The spectral bound $s(A)$ is defined by

$$s(A) = \sup\{\operatorname{Re} \lambda : \lambda \in \sigma(A)\}$$

and the growth bound $\omega(T_t)$ of (T_t) is given by

$$\omega = \inf\left\{w : \sup_{t>0} e^{-wt} \|T_t\| < \infty\right\}.$$

If $\dim X < \infty$ the classical result of Liapunov states that $s(A) = \omega(T_t)$ and this implies that the solutions $u(t) = T_t x_0$ of the Cauchy problem

$$u'(t) = Au(t), \quad u(0) = x_0$$

are asymptotically stable if $s(A) < 0$. If $\dim X = \infty$ it may happen that $s(A) < \omega(T_t)$, even if X is a Hilbert space or T_t is a positive semigroup on a Banach function space (see, e.g., [8, A-IV.1]). On the other hand, it is known that $s(A) = \omega(T_t)$ if (T_t) is a positive semigroup on one of the spaces $L_2(\Omega, m)$, $L_1(\Omega, m)$, or $C(K)$ (see [8, C-IV 1.1], [3, 9.1], and the references to the original papers of Gearhart, Derndinger, Greiner, and Nagel given there), where (Ω, m) is a σ -finite measure space.

Apparently, the question whether every positive semigroup T_t on a space $L_p(\Omega, m)$ satisfies $\omega(T_t) = s(A)$ remained an open problem (cf. [8, C IV.1.1], [3, 9.1], [1]) although there are some partial results under additional assumptions on T_t ; e.g., it is already known that $s(A) = \omega(T_t)$ if (T_t) is an analytical semigroup ([8, A-IV 1.14]), if the semigroup operates on all $L_q(\Omega, m)$ spaces

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for $1 \leq q \leq \infty$ ([10], [8, C-IV-1.1]), if the resolvent of A satisfies some additional growth estimates [6], or if A is a differential operator with constant, real coefficients ([4]).

In this paper we prove that no additional assumptions are needed:

Theorem. *Let (T_t) be a positive c_0 -semigroup on $L_p(\Omega, m)$, $1 \leq p < \infty$, with generator A . Then the spectral bound $s(A)$ equals the growth bound $\omega(T_t)$.*

In particular, in this situation $s(A) < 0$ implies that all mild solutions of the Cauchy problem (1) are asymptotically stable. A partial result in this direction that does not depend on positivity is contained in [9].

We prove the theorem in §2 using a recent spectral mapping theorem of Latushkin and Montgomery-Smith ([7], see Lemma 1 below) and an extrapolation theorem for positive operators on L_p ([11], see Lemma 2 below).

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2. PROOF OF THE THEOREM

Our proof uses the following recent result of Latushkin and Montgomery-Smith.

Lemma 1 ([7]). *Let T_t be a c_0 -semigroup on a Banach space X . For $1 \leq p < \infty$ put $Y = L_p(\mathbb{R}, X)$ and define the semigroup S_t on Y by*

$$S_t f(s) = T_t(f(s-t)), \quad s \in \mathbb{R},$$

for $f \in Y$. Then the spectral bound $s(B)$ of the generator B of S_t equals the growth bound of (T_t) on X .

Proof. Combine 2.5 of [7] and [8, A IV 1.11]. $\square$

According to Lemma 1, it remains to show that a positive semigroup T_t on $X = L_p(\Omega, m)$ with generator A satisfies $s(B) \leq s(A)$. This can be shown directly for $p = 1$ (see Lemma 3 below) and the case of general p will be reduced to $p = 1$ with the help of the following extrapolation result.

Lemma 2 ([11, Theorem 2.1]). *Let R be a positive linear operator on $L_p(\Omega, m)$, $1 \leq p < \infty$.*

Then there is a density g with $0 < g \in L_1(\Omega, m)$ such that the operator $\rho R \rho^{-1}$ extends to a bounded linear operator on all spaces $L_q(\Omega, g dm)$, $1 \leq q \leq \infty$, where $\rho : L_p(\Omega, m) \rightarrow L_p(\Omega, g dm)$ is the isometry $\rho(f) = f \cdot g^{-1/p}$. Furthermore, for all $1 \leq q \leq \infty$

$$\|\rho R \rho^{-1}\|_{L_q(\Omega, g dm)} \leq 2\|R\|_{L_p(\Omega, m)}.$$

Lemma 3. *Let T_t be a positive semigroup on $L = L_1(\Omega, m)$ with generator A . Put $Y = L_1(\mathbb{R}, L)$ and let B be the generator of the semigroup $(S_t f)(s) = T_t(f(s-t))$ on Y . Then $s(B) = s(A)$ and for all λ with $\lambda > s(A)$*

$$\|R(\lambda, B)\|_Y \leq \|R(\lambda, A)\|_L.$$

This is also true if B denotes the generator of the semigroup $(S_t f)(s) = T_t(f(s+t))$.

Proof. By [3, 7.4], the integral $\int_0^\infty e^{-\lambda t} T_t x dt$ exists as norm convergent improper Bochner integral for all $\lambda > s(A)$ and $x \in X_+$. By Fubini's theorem we

get for all $T < \infty$:

$$\int_0^T e^{-\lambda t} \|T_t x\|_L dt = \int_{\Omega} \left[\int_0^T e^{-\lambda t} T_t(x) dt \right] (w) dm(w) \leq \|R(\lambda, A)x\|_L.$$

For $f \in Y_+$ and a fixed $t > 0$ we have

$$\begin{aligned} \|S_t f\|_Y &= \int_{-\infty}^{\infty} \|T_t f(s-t)\|_L ds = \left\| \int_{-\infty}^{\infty} T_t(f(s-t)) ds \right\|_L \\ &= \left\| T_t \left(\int_{-\infty}^{\infty} f(s-t) dt \right) \right\| = \|T_t(F)\| \end{aligned}$$

where $F = \int_{-\infty}^{\infty} f(s) ds$ is a Bochner integral in L with $\|F\|_L = \|f\|_Y$. With the first estimate we get for $\lambda > s(A)$

$$\begin{aligned} \|R(\lambda, B)f\|_Y &= \left\| \int_0^{\infty} e^{-\lambda t} S_t f dt \right\|_Y \leq \int_0^{\infty} e^{-\lambda t} \|T_t(F)\| dt \\ &= \|R(\lambda, A)F\| \leq \|R(\lambda, A)\|_L \cdot \|f\|. \end{aligned}$$

Since $R(\lambda, B)$ is positive, we have, for general $f \in Y$, $\|R(\lambda, B)f\| \leq \|R(\lambda, B)|f|\|$, and the first claim follows. This proof can easily be adopted to the case $(S_t f)(s) = T_t(f(s+t))$. $\square$

Proof of the theorem. Step 1: An extrapolation argument. A moment's reflection shows that it is sufficient to prove that $s(A) < 0$ implies that $\omega(T_t) \leq 0$. Using the extrapolation Lemma 2 we can assume that (Ω, m) is finite and that the positive operator $R(0, A)$ on $L_p(\Omega, m)$ extend to bounded linear operators $R_q(0)$ on all $L_q(\Omega, m)$, $1 \leq q \leq \infty$, in the sense that $R_q(0)f = R_p(0, A)f$ for $f \in L_q(\Omega, m) \cap L_p(\Omega, m)$. Furthermore,

$$(1) \quad \|R_q(0)\| \leq 2\|R(0, A)\|.$$

For $\bar{R}_q(0) : L_q(\Omega, m) \rightarrow L_q(\Omega, m)$ defined by $\bar{R}_q(0) = R_{q'}(0)^*$, $\frac{1}{p} + \frac{1}{q'} = 1$, $1 \leq q \leq \infty$, we have

$$\bar{R}_q(0)f = R(0, A^*)f \quad \text{for } f \in L_q(\Omega, m) \cap L_{p'}(\Omega, m),$$

where A^* is the generator of T_t^* on $L_{p'}(\Omega, m)$, $\frac{1}{p} + \frac{1}{p'} = 1$. Indeed, for all $f \in L_q \cap L_{p'}$ and $g \in L_{q'} \cap L_p$ we have $\langle g, \bar{R}_q(0)f \rangle = \langle g, R(0, A^*)f \rangle$ if and only if $\langle R_{q'}(0)g, f \rangle = \langle R(0, A)g, f \rangle$. Since T_t is a positive semigroup, it is well known that for all $\lambda \geq 0$

$$0 \leq R(\lambda, A) \leq R(0, A), \quad 0 \leq R(\lambda, A^*) \leq R(0, A^*)$$

and for all λ with $\operatorname{Re} \lambda \geq 0$

$$|R(\lambda, A)| \leq R(0, A), \quad |R(\lambda, A^*)| \leq R(0, A^*)$$

where $|S|$ is the modulus of a regular operator S on L_p (see, e.g., [7]). Hence there are extensions $R_q(\lambda) : L_q(\Omega, m) \rightarrow L_q(\Omega, m)$ of $R(\lambda, A)$ and $\bar{R}_q(\lambda) : L_q(\Omega, m) \rightarrow L_q(\Omega, m)$ of $R(\lambda, A^*)$ for all $1 \leq q \leq \infty$ and $\operatorname{Re} \lambda \geq 0$, such that the functions $\lambda \rightarrow R_q(\lambda)$, $\lambda \rightarrow \bar{R}_q(\lambda)$ satisfy the resolvent equation on $\{\lambda : \operatorname{Re} \lambda \geq 0\}$. But $R_1(\lambda)$ may not be the resolvent of a positive semigroup on $L_1(\Omega, m)$. To get around this we use the Yosida approximations

$$\begin{aligned} A_\mu &= \mu^2 R(\mu, A) - \mu I, & T_t^\mu &= \exp(tA_\mu), \\ \bar{A}_\mu &= \mu^2 R(\mu, A^*) - \mu I, & \bar{T}_t^\mu &= \exp(t\bar{A}_\mu) \end{aligned}$$

($\mu > 0, t \geq 0$) of A on $L_p(\Omega, m)$ and A^* on $L_{p'}(\Omega, m)$. These operators satisfy:

- (i) $s(A_\mu) < 0$ and for $\operatorname{Re} \lambda \geq 0$ we have $R(\lambda, A_\mu) = \frac{1}{\mu + \lambda} + (\frac{\mu}{\lambda + \mu})^2 R(\frac{\lambda\mu}{\mu + \lambda}, A)$.
(ii) For $\omega > \omega(T_t)$ there is a constant $D < \infty$ such that for $\mu \geq 2\omega$

$$(2) \quad \|T_t^\mu\|_p \leq De^{2\omega t}.$$

Analogous statements hold for $\bar{A}_\mu$ and $\bar{T}_t^\mu$ on $L_{p'}(\Omega, m)$. Indeed, (i) holds since the resolvent equation for $R(\lambda, A)$ implies

$$\begin{aligned} (\lambda I - A_\mu) \left[\frac{1}{\mu + \lambda} I + \left(\frac{\mu}{\mu + \lambda} \right)^2 R \left(\frac{\lambda\mu}{\mu + \lambda}, A \right) \right] \\ = I + \frac{\mu^2}{\mu + \lambda} \left[R \left(\frac{\lambda\mu}{\mu + \lambda}, A \right) - R(\mu, A) - \frac{\mu^2}{\mu + \lambda} R \left(\frac{\lambda\mu}{\mu + \lambda}, A \right) R(\mu, A) \right] \\ = I. \end{aligned}$$

To see (ii) choose a constant $D < \infty$ with

$$\|T_t\| \leq De^{\omega t}, \quad (\lambda - \omega)^n \|R(\lambda, A)^n\|_{L_p} \leq D$$

for all $t \geq 0, \lambda > \omega$, and $n \in \mathbb{N}$. From the definition of A_μ and T_t^μ we get for $\mu \geq 2\omega$

$$\begin{aligned} \|T_t^\mu\|_{L_p} &\leq e^{-\mu t} \sum_{n=0}^{\infty} \frac{1}{n!} t^n \mu^{2n} \|R(\mu, A)^n\| \\ &\leq e^{-\mu t} \sum_{n=0}^{\infty} \frac{1}{n!} t^n \left(\frac{\mu^2}{\mu - \omega} \right)^n D \leq D \exp \left(\frac{\mu\omega}{\mu - \omega} t \right) \leq De^{2\omega t}. \end{aligned}$$

By replacing $R(\mu, A)$ by $R_q(\mu)$ and $R(\mu, A^*)$ by $\bar{R}_q(\mu)$ we can extend the operators $A_\mu, \bar{A}_\mu$ and $T_\mu, \bar{T}_t^\mu$ to bounded operators $A_{\mu,q}, \bar{A}_{\mu,q}$ and $T_t^{\mu,q}, \bar{T}_t^{\mu,q}$ on $L_q(\Omega, m)$ for all $1 \leq q \leq \infty$. Since the proof of (i) only used the resolvent equation, we get in the same way for $q = 1$

$$(3) \quad \begin{aligned} s(A_{\mu,1}) &< 0, \quad R(0, A_{\mu,1}) = \frac{1}{\mu} + R_1(0), \\ s(\bar{A}_{\mu,1}) &< 0, \quad R(0, \bar{A}_{\mu,1}) = \frac{1}{\mu} + \bar{R}_1(0). \end{aligned}$$

For simplicity we now drop the index q and distinguish the various extensions by context.

Step 2: Resolvent-estimates for the evolutionary semigroups. Consider now $Y_q = L_q(\mathbb{R}, L_q(\Omega, m))$, $1 \leq q \leq \infty$, and define for $\mu > 0, t \geq 0$, and $f \in Y_q$

$$\begin{aligned} S_t^\mu f(s) &= T_t^\mu(f(s-t)), \quad s \in \mathbb{R}, \\ \bar{S}_t^\mu f(s) &= \bar{T}_t^\mu(f(s+t)), \quad s \in \mathbb{R}. \end{aligned}$$

For every $\mu > 0, (S_t^\mu)_{t \geq 0}$ and $(\bar{S}_t^\mu)_{t \geq 0}$ define positive c_0 -semigroups on Y_q , $1 \leq q < \infty$, and w^* -semigroups on Y_∞ with generators

$$\begin{aligned} (B_\mu f)(s) &= -f'(s) + A_\mu(f(s)), \quad s \in \mathbb{R}, \\ (\bar{B}_\mu f)(s) &= +f'(s) + \bar{A}_\mu(f(s)), \quad s \in \mathbb{R}, \end{aligned}$$

with $D(B_\mu) = D(\overline{B}_\mu) = \{f \in Y_q : f' \text{ exists, a.e., and } f' \in Y_q\}$, respectively. If we think of Y_∞ as a subspace of Y_1^* , then $(\overline{S}_t^\mu)^*$, the dual of $\overline{S}_t^\mu : Y_1 \rightarrow Y_1$ in Y_1^* , is an extension of $S_t^\mu : Y_\infty \rightarrow Y_\infty$. Furthermore, $(\overline{B}_\mu)^*$ is the weak- w^* -generator of $(\overline{S}_t^\mu)^*$ and B_μ is the part of $(\overline{B}_\mu)^*$ in Y_∞ . Hence

$$(4) \quad R(\lambda, B_\mu) = R(\lambda, \overline{B}_\mu)^*|_{Y_\infty}$$

for all $\lambda \in \rho(\overline{B}_\mu|_{Y_1})$ (see, e.g., [5, 14.3]).

We now apply Lemma 3 to each semigroup T_t^μ and $\overline{T}_t^\mu$ on $L = L_1(\Omega, m)$ and obtain for $\operatorname{Re} \lambda \geq 0$

$$\begin{aligned} \|R(\lambda, B_\mu)\|_{Y_1} &\leq \|R(0, B_\mu)\|_{Y_1} \leq \|R(0, A_\mu)\|_L, \\ \|R(\lambda, \overline{B}_\mu)\|_{Y_1} &\leq \|R(0, \overline{B}_\mu)\|_{Y_1} \leq \|R(0, \overline{A}_\mu)\|_L. \end{aligned}$$

From (1), (3), and (4) we obtain for $\mu \geq 1$, $\operatorname{Re} \lambda \geq 0$

$$\begin{aligned} \|R(\lambda, B_\mu)\|_{Y_1} &\leq 1 + 2\|R(0, A)\|_{L_p} = C, \\ \|R(\lambda, B_\mu)\|_{Y_\infty} &\leq \|R(\lambda, \overline{B}_\mu)\|_{Y_1} \leq 1 + 2\|R(0, A^*)\|_{L_p'} = C. \end{aligned}$$

Since A_μ is a bounded operator on $L_q(\Omega, m)$, we have $s(B_\mu|_{Y_q}) = s(A_\mu|_{L_q}) < 0$ for all $1 \leq q \leq \infty$. For the complex interpolation method we have $[Y_\infty, Y_1]_{1/p} = Y_p$. This follows from [2, Theorem 5.1.2], the inclusions $L_\infty^0(\mathbb{R}, L_\infty(\Omega)) \subset Y_\infty \subset L_\infty(\mathbb{R} \times \Omega)$, and the identities $Y_p \simeq L_p(\mathbb{R} \times \Omega)$ for $1 \leq p < \infty$. Interpolating in the scale Y_q we obtain for all $\mu > 1$ and $\operatorname{Re} \lambda \geq 0$ that

$$(5) \quad \|R(\lambda, B_\mu)\|_{Y_p} \leq C.$$

Step 3: The spectral bound of B_μ . Now we want to show that $R(\lambda, B_\mu)$ converges for $\mu \rightarrow \infty$ to the resolvent of the c_0 -semigroup

$$(S_t f)(s) = T_t(f(s-t)), \quad s \in \mathbb{R},$$

on Y_p , whose generator B is the closure of

$$B_0 f(s) = -f'(s) + A(f(s)), \quad s \in \mathbb{R}.$$

$$D(B_0) = \{f \in Y_p : f \in L_p(D(A)) \text{ and } f' \text{ exists, a.e., with } f' \in Y_p\}.$$

For $\mu \geq 2w$ we therefore get from (2) that

$$(6) \quad \|T_t^\mu\|, \|S_t^\mu\| \leq De^{2wt}.$$

Hence, for a λ with $\operatorname{Re} \lambda \geq 3w$ and $f \in Y_p$ the Bochner integral

$$(R(\lambda, B)f)(\cdot) = \int_0^\infty e^{-\lambda t} T_t(f(\cdot - t)) dt$$

exists in Y_p and with (6)

$$\begin{aligned} &\|R(\lambda, B)f - R(\lambda, B_\mu)f\|_{Y_p} \\ (7) \quad &\leq \int_0^\infty \left(\int_{-\infty}^\infty [e^{-\lambda t} \|T_t(f(s-t)) - T_t^\mu f(s-t)\|_{L_p}]^p ds \right)^{1/p} dt \\ &\leq \int_0^\infty e^{-wt} \left(\int_{-\infty}^\infty \|e^{-2wt} T_t(f(s)) - e^{-2wt} T_t^\mu(f(s))\|^p ds \right)^{1/p} dt \\ &\rightarrow 0 \end{aligned}$$

for $\mu \rightarrow \infty$ by Lebesgue's convergence theorem since $T_t^\mu(x) \rightarrow T_t(x)$ for all $t \geq 0$ and $x \in L_p(\Omega, m)$ (see, e.g., [5, 12.3]).

By (5) and Vitali's theorem ([5, 3.14.1]) the limits

$$R_\lambda f = \lim_{\mu \rightarrow \infty} R(\lambda, B_\mu)f, \quad f \in Y_p,$$

exist for all λ with $\operatorname{Re} \lambda > 0$ and define an operator function $\lambda \in \mathbb{C}^+ \rightarrow R_\lambda \in B(Y_p)$ with $\|R_\lambda\| \leq C$ and satisfying the resolvent equation. By (7) $R_\lambda = R(\lambda, B)$ for $\operatorname{Re} \lambda > w$ and since R_λ satisfies the resolvent equation, we have $\operatorname{Im} R_\lambda = D(B)$, $\operatorname{Ker} R_\lambda = \{0\}$ for all $\operatorname{Re} \lambda > 0$. By analytic continuation $R_\lambda = R(\lambda, B)$ for all λ with $\operatorname{Re} \lambda > 0$.

In particular, $s(B|Y_p) \leq 0$ and we can appeal to Lemma 1 to get $\omega(T_i) \leq 0$. $\square$

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