SYMMETRIC ORTHOGONAL POLYNOMIALS AND THE ASSOCIATED ORTHOGONAL L-POLYNOMIALS

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(Communicated by Hal L. Smith)

ABSTRACT. We show how symmetric orthogonal polynomials can be linked to polynomials associated with certain orthogonal L-polynomials. We provide some examples to illustrate the results obtained. Finally as an application, we derive information regarding the orthogonal polynomials associated with the weight function $(1 + kx^2)(1 - x^2)^{-1/2}$, $k > 0$.

1. INTRODUCTION

Let $Y(n)$ be the space of all real monic polynomials $P_n(x)$ of degree n that satisfy the symmetric property $P_n(x) = (-1)^n P_n(-x)$. Let $Z(n, \beta)$ be the space of all real monic polynomials $Q_n(t)$ of degree n that satisfy the rather different symmetric property $Q_n(t) = t^n Q_n(\beta^2/t)/(-\beta)^n$.

It is well known (see [1, 13]) that if $w(x)$ is a weight function on $(-d, d)$, where $0 < d \leq \infty$, such that $w(x) = w(-x)$, then the associated monic orthogonal polynomials $B_n(w; x)$, $n \geq 0$, are such that $B_n(w; x) \in Y(n)$. Furthermore, they satisfy

$$(1.1) \quad B_{n+1}(w; x) = xB_n(w; x) - \alpha_{n+1}(w)B_{n-1}(w; x), \quad n \geq 1,$$

where

$$\alpha_{n+1}(w) = \frac{\int_{-d}^d x^n B_n(w; x) w(x) dx}{\int_{-d}^d x^{n-1} B_{n-1}(w; x) w(x) dx} > 0, \quad n \geq 1.$$

Now, some lesser known results are the following. Let $\nu(t)$ be a strong weight function defined on $(\beta^2/b, b)$, where $0 < \beta < b \leq \infty$, such that

$$(1.2) \quad \sqrt{t} \nu(t) = \sqrt{\beta^2/t} \nu(\beta^2/t).$$

Then the monic polynomials $\tilde{B}_n(\nu; t)$, $n \geq 0$, defined uniquely by

$$(1.3) \quad \int_{\beta^2/b}^b t^{-n+s} \tilde{B}_n(\nu; t) \nu(t) dt = 0, \quad 0 \leq s \leq n-1,$$

Received by the editors March 29, 1994.

1991 *Mathematics Subject Classification.* Primary 33C45, 42C05.

Key words and phrases. Orthogonal polynomials, L-polynomials, recurrence relations.

This work was supported by the Brazilian research foundations CNPq and FAPESP.

are such that $\tilde{B}_n(\nu; t) \in \mathbf{Z}(n, \beta)$ and satisfy

$$(1.4) \quad \tilde{B}_{n+1}(\nu; t) = (t - \beta)\tilde{B}_n(\nu; t) - \tilde{\alpha}_{n+1}(\nu)t\tilde{B}_{n-1}(\nu; t), \quad n \geq 1,$$

where

$$\tilde{\alpha}_{n+1}(\nu) = \frac{\int_{\beta^2/b}^b \tilde{B}_n(\nu; t)\nu(t) dt}{\int_{\beta^2/b}^b \tilde{B}_{n-1}(\nu; t)\nu(t) dt} > 0, \quad n \geq 1.$$

For proofs, and other information on these results, see [8, 12]. In [9, 10, 11, 12] some results can be found regarding the polynomials defined by (1.3) for weight functions satisfying a property different from (1.2). For some studies of polynomials given by a variation of (1.3), see [5, 6, 7].

It is easily verified from (1.3) that the Laurent polynomials or L-polynomials given by

$$R_{2m}(t) = t^{-m}\tilde{B}_{2m}(\nu; t), \quad R_{2m+1}(t) = t^{-m-1}\tilde{B}_{2m+1}(\nu; t), \quad m \geq 0,$$

form a sequence of orthogonal L-polynomials on $(\beta^2/b, b)$ in relation to the strong weight function $\nu(t)$. For information on orthogonal L-polynomials and polynomials similar to $\tilde{B}_n(\nu; t)$, but where $\nu(t)$ is any strong weight function, we refer to [2, 3, 4].

In this article we show the relation between the polynomials $B_n(w; x)$ and $\tilde{B}_n(\nu; t)$, when the associated weight functions are linked in a certain way. We also provide some examples to illustrate the results obtained. Finally as an application, we derive information regarding the orthogonal polynomials $B_n(W; x)$, $n \geq 0$, where $d = 1$ and $W(x) = (1 + kx^2)(1 - x^2)^{-1/2}$, for $k > 0$.

2. PRELIMINARY RESULTS

Throughout the rest of this article we assume $\alpha > 0$ and $\beta > 0$.

Theorem 2.1. *Let the sequence of polynomials $\{P_n(x)\}$ be such that $P_n(x) \in \mathbf{Y}(n)$. Then*

$$P_n(x) = B_n(w; x), \quad n \geq 0,$$

if and only if

$$(2.1) \quad \int_{-d}^d \left\{ \sqrt{\alpha x^2 + \beta} + \sqrt{\alpha}x \right\}^{-(n-1)+2s} \frac{P_n(x)}{\sqrt{\alpha x^2 + \beta}} w(x) dx = 0, \quad 0 \leq s \leq n-1.$$

Proof. Since $\{\sqrt{\alpha x^2 + \beta} + \sqrt{\alpha}x\}^{-1} = \{\sqrt{\alpha x^2 + \beta} - \sqrt{\alpha}x\}/\beta$, (2.1) is equivalent to

$$\int_{-d}^d \left\{ \sqrt{\alpha x^2 + \beta} \pm \sqrt{\alpha}x \right\}^{2l} \frac{P_{2m+1}(x)}{\sqrt{\alpha x^2 + \beta}} w(x) dx = 0$$

and

$$\int_{-d}^d \left\{ \sqrt{\alpha x^2 + \beta} \pm \sqrt{\alpha}x \right\}^{2l+1} \frac{P_{2m+2}(x)}{\sqrt{\alpha x^2 + \beta}} w(x) dx = 0,$$

for $l = 0, 1, \dots, m$ and $m \geq 0$.

Using the symmetric property of $P_n(x)$, this can also be given in the equivalent form

$$(2.2) \quad \int_{-d}^d \left[\sum_{r=0}^{l-1} \binom{2l}{2r+1} \{\alpha x^2 + \beta\}^{l-r-1} \{\pm \sqrt{\alpha}x\}^{2r+1} \right] P_{2m+1}(x)w(x) dx = 0,$$

$$\int_{-d}^d \left[\sum_{r=0}^l \binom{2l+1}{2r} \{\alpha x^2 + \beta\}^{l-r} \{\pm \sqrt{\alpha}x\}^{2r} \right] P_{2m+2}(x)w(x) dx = 0,$$

for $l = 0, 1, \dots, m$ and $m \geq 0$.

Now, if $P_n(x) = B_n(w; x)$ for $n \geq 0$, then (2.2) must hold as $B_n(w; x)$ is orthogonal to all polynomials of degree $\leq n$.

Conversely, if (2.2) holds, then, for example, $P_{2m+1}(x)$ is orthogonal to the odd polynomials

$$\sum_{r=0}^{l-1} \binom{2l}{2r+1} \{\alpha x^2 + \beta\}^{l-r-1} \{\sqrt{\alpha}x\}^{2r+1},$$

of "precise" degree $2l-1$, for $l = 0, 1, \dots, m$. Since these polynomials must form a basis for all odd polynomials of degree $\leq 2m-1$, $P_{2m+1}(x)$ must be orthogonal to all odd polynomials of degree $\leq 2m-1$. Hence, as $P_{2m+1}(x)$ is monic and is also odd, we obtain $P_{2m+1}(x) = B_{2m+1}(w; x)$. In a similar way we also show that $P_{2m}(x) = B_{2m}(w; x)$. This completes the proof of the theorem.

We now consider the transformation

$$t(x) = \left\{ \sqrt{\alpha x^2 + \beta} + \sqrt{\alpha}x \right\}^2, \quad x \in (-\infty, \infty).$$

The transformation $t(x)$ represents a one-to-one correspondence between $(-\infty, \infty)$ and $(0, \infty)$. The inverse of $t(x)$ is

$$(2.3) \quad x(t) = \frac{1}{2\sqrt{\alpha}} \left(\sqrt{t} - \beta/\sqrt{t} \right), \quad t \in (0, \infty).$$

If $x = d$ corresponds to $t = b$, that is, if $\sqrt{b} = \sqrt{\alpha d^2 + \beta} + \sqrt{\alpha}d$, then it is easily seen that $x = -d$ corresponds to $t = \beta^2/b$.

Theorem 2.2. With $x(t)$ given by (2.3), let $Q(t) = (2\sqrt{\alpha}t)^n P(x(t))$. Then

$$P(x) \in \mathbf{Y}(n) \quad \text{iff} \quad Q(t) \in \mathbf{Z}(n, \beta).$$

Proof. Since $x(\beta^2/t) = -x(t)$, it follows that

$$\frac{t^n Q(\beta^2/t)}{(-\beta)^n} = (-1)^n (2\sqrt{\alpha}t)^n P(-x(t)).$$

Hence, $P(x) = (-1)^n P(-x)$ iff $Q(t) = t^n Q(\beta^2/t)/(-\beta)^n$. Thus, all we need to show is that if $P(x)$ is a monic polynomial of degree n , then so is $Q(t)$ and vice versa. This can be established using the following observations.

If $Q(t) \in \mathbf{Z}(n, \beta)$ and if $Q(t) = \sum_{r=0}^n q_r t^r$, where $q_n = 1$, then $q_{n-r} = (-\beta)^{2r-n} q_r$. Hence, for $r = [(n+1)/2], \dots, n$

$$\{2\sqrt{\alpha t(x)}\}^{-n} [q_r \{t(x)\}^n + q_{n-r} \{t(x)\}^{n-r}]$$

is a polynomial of the form

$$(4\alpha)^{r-n} q_r x^{2r-n} + \text{lower degrees.}$$

On the other hand, if $P(x) \in Y(n)$, then for $n = 2m$ we can write $P(x) = \sum_{r=0}^m p_{2r} x^{2r}$, where $p_{2m} = 1$. Hence, $(2\sqrt{\alpha t})^{2m} p_{2r} \{x(t)\}^{2r}$ is the polynomial

$$(4\alpha)^{m-r} p_{2r} t^{m-r} (t - \beta)^{2r},$$

for $r = 0, 1, \dots, m$. Similarly for $n = 2m + 1$, we can write $P(x) = \sum_{r=0}^m p_{2r+1} x^{2r+1}$, where $p_{2m+1} = 1$. Hence, $(2\sqrt{\alpha t})^{2m+1} p_{2r+1} \{x(t)\}^{2r+1}$ is the polynomial

$$(4\alpha)^{m-r} p_{2r+1} t^{m-r} (t - \beta)^{2r}, \quad r = 0, 1, \dots, m.$$

3. THE MAIN RESULTS

We first consider a result regarding the weight functions.

Theorem 3.1. *Let b and d be such that $\sqrt{b} = \sqrt{\alpha d^2 + \beta} + \sqrt{\alpha d}$ and let*

$$V(t) = At^{-1/2} W(x(t)).$$

Then $W(x)$ is a weight function on $(-d, d)$ such that $W(x) = W(-x)$ if and only if $V(t)$ is a strong weight function on $(\beta^2/b, b)$ such that $\sqrt{t}V(t) = \sqrt{\beta^2/t}V(\beta^2/t)$. Here, A is any positive number.

Proof. It is easily verified that if $W(x)$ is positive and satisfies $W(x) = W(-x)$ in $(-d, d)$, then $V(t)$ is positive and satisfies $\sqrt{t}V(t) = \sqrt{\beta^2/t}V(\beta^2/t)$ in $(\beta^2/b, b)$ and vice versa. Therefore we need only to prove that when the moments (including the negative ones for $V(t)$) exist and are finite for one of the functions, the same is also true for the other function. We have

$$\int_{\beta^2/b}^b t^r V(t) dt = A 2\sqrt{\alpha} \int_{-d}^d \left\{ \sqrt{\alpha x^2 + \beta} + \sqrt{\alpha x} \right\}^{2r+1} \frac{W(x)}{\sqrt{\alpha x^2 + \beta}} dx.$$

Since $W(x) = W(-x)$, this can be written as

$$\int_{\beta^2/b}^b t^r V(t) dt = \int_{-d}^d S_r(x) W(x) dx,$$

where $S_r(x)$ is an even polynomial of exact degree $2r$ for $r \geq 0$ and is an even polynomial of exact degree $|2r| - 2$ for $r \leq -1$. As the monomials x^{2r} and the polynomials $S_r(x)$ can be expressed as linear combinations of each other, the proof of the theorem follows.

Theorem 3.2. *Let $W(x)$ and $V(t)$ be a pair of weight functions given by Theorem 3.1. Then for $n \geq 0$*

$$(3.1) \quad \tilde{B}_n(V; t) = (2\sqrt{\alpha t})^n B_n(W; x(t)).$$

Proof. From Theorems 2.1 and 3.1 it follows that

$$\int_{\beta^2/b}^b t^{-n+s} (2\sqrt{\alpha t})^n B_n(W; x(t)) V(t) dt = 0, \quad 0 \leq s \leq n-1.$$

Hence from (1.3) and Theorem 2.2, the result of the theorem is immediate.

As a consequence of (3.1) we obtain from (1.1) and (1.4) that

$$(3.2) \quad \tilde{\alpha}_{n+1}(V) = 4\alpha\alpha_{n+1}(W), \quad n \geq 1.$$

Furthermore, if $G(x, u)$ is a generating function for the polynomials $B_n(W; x)$, then

$$(3.3) \quad F(t, u) = G(x(t), 2\sqrt{\alpha}tu)$$

is a generating function for $\tilde{B}_n(V; t)$.

4. ILLUSTRATIVE EXAMPLES

Example 1. The Tchebyshev case. If $W(x) = 1/\sqrt{1-x^2}$ in $(-1, 1)$, then from Theorem 3.1

$$V(t) = \frac{1}{\sqrt{b-t}\sqrt{t-a}} \quad \text{in } (a, b),$$

where $\sqrt{b} = \sqrt{\alpha + \beta} + \sqrt{\alpha}$ and $\sqrt{a} = \sqrt{\alpha + \beta} - \sqrt{\alpha}$.

The associated polynomials $\tilde{B}_n(V; t)$, $n \geq 1$, and their interpolatory quadrature rules have already been studied in [8]. From [8], we can also confirm that (3.2) holds.

Now from (3.3) we get

$$F(t, u) = \frac{1 - (t - \beta)u}{1 - 2(t - \beta)u + 4\alpha tu^2},$$

as a generating function for $\tilde{B}_n(V; t)$.

Example 2. The Gegenbauer case. For $\lambda > 0$, if $W(x) = (1 - x^2)^{\lambda-1/2}$ in $(-1, 1)$, then

$$V(t) = t^{-\lambda}(b-t)^{\lambda-1/2}(t-a)^{\lambda-1/2} \quad \text{in } (a, b),$$

where $\sqrt{b} = \sqrt{\alpha + \beta} + \sqrt{\alpha}$ and $\sqrt{a} = \sqrt{\alpha + \beta} - \sqrt{\alpha}$. From the recurrence relation for the monic Gegenbauer polynomials it then follows that

$$\tilde{\alpha}_{n+1}(V) = \frac{n(n+2\lambda-1)}{(n+\lambda)(n+\lambda-1)}\alpha, \quad n \geq 1.$$

For $\lambda = 1/2$, that is, for the Legendre case, this result has already been given in [8].

From (3.3), we obtain as a generating function for $\tilde{B}_n(V; t)$,

$$F(t, u) = \{1 - 2(t - \beta)u + 4\alpha tu^2\}^{-\lambda}.$$

Example 3. The Hermite case. If $W(x) = \exp(-x^2)$ in $(-\infty, \infty)$, then

$$V(t) = t^{-1/2} \exp\left(-\frac{(t + \beta^2/t)}{4\alpha}\right) \quad \text{in } (0, \infty).$$

From the recurrence relation for the monic Hermite polynomials we find

$$\tilde{\alpha}_{n+1}(V) = 2\alpha n, \quad n \geq 1.$$

This result for $\alpha = 1/2$ has also been given in [8].

Using (3.3) as a generating function for $\tilde{B}_n(V; t)$ we have

$$F(t, u) = \exp(2(t - \beta)u - 4\alpha tu^2).$$

5. AN APPLICATION

Let $W(x) = (1 + kx^2)(1 - x^2)^{-1/2}$ in $(-1, 1)$, where $k > 0$. Choosing α and β such that $k = \alpha/\beta$, we obtain

$$V(t) = \frac{t^{-1}(t + \beta)^2}{\sqrt{b - t}\sqrt{t - a}} = \frac{t(1 + \beta/t)^2}{\sqrt{b - t}\sqrt{t - a}} \quad \text{in } (a, b),$$

where $\sqrt{b} = \sqrt{\alpha + \beta} + \sqrt{\alpha}$ and $\sqrt{a} = \sqrt{\alpha + \beta} - \sqrt{\alpha}$. Equivalently, we can also write $\beta = \sqrt{ab}$ and $\sqrt{\alpha} = (\sqrt{b} - \sqrt{a})/2$.

It has been shown in [12] that the polynomials $\tilde{B}_n(V; t)$ satisfy the recurrence relation (1.4) with

$$\tilde{\alpha}_{n+1}(V) = \frac{K_{n-1}K_{n+2}}{K_nK_{n+1}}\alpha, \quad n \geq 1,$$

with $K_n = (1 + l)^n + (1 - l)^n$, $n \geq 0$, and $l = \sqrt{1 + \alpha/\beta}$. Hence from (3.2) we obtain the following result.

Theorem 5.1. *The monic orthogonal polynomials $B_n(W; x)$, $n \geq 0$, associated with the weight function*

$$W(x) = \frac{1 + kx^2}{\sqrt{1 - x^2}} \quad \text{in } (-1, 1),$$

where $k > 0$, satisfy

$$(5.1) \quad B_{n+1}(W; x) = xB_n(W; x) - \frac{1}{4} \frac{K_{n-1}K_{n+2}}{K_nK_{n+1}} B_{n-1}(W; x), \quad n \geq 1,$$

with $B_0(W; x) = 1$, $B_1(W; x) = x$. Here $K_n = (1 + l)^n + (1 - l)^n$, $n \geq 0$, and $l = \sqrt{1 + k}$.

Since $\int_{-1}^1 W(x) dx = (2 + k)\pi/2$, we obtain from (5.1) the orthogonality property

$$\int_{-1}^1 B_m(W; x) B_n(W; x) \frac{1 + kx^2}{\sqrt{1 - x^2}} dx = \delta_{mn} 2^{-2n} \frac{K_{n+2}}{K_n} \pi/2.$$

Finally, if we let $P_n(W; x) = 2^{n-1} K_n B_n(W; x)$ for $n \geq 0$, then

$$\int_{-1}^1 P_m(W; x) P_n(W; x) \frac{1 + kx^2}{\sqrt{1 - x^2}} dx = \delta_{mn} K_n K_{n+2} \pi/8$$

and

$$K_n P_{n+1}(W; x) = 2K_{n+1} x P_n(W; x) - K_{n+2} P_{n-1}(W; x), \quad n \geq 1,$$

with $P_0(W; x) = 1$ and $P_1(W; x) = 2x$.

REFERENCES

1. T. S. Chihara, *An introduction to orthogonal polynomials*, Math. Appl., Gordon and Breach, New York, 1978.
2. W. B. Jones, O. Njåstad, and W. J. Thron, *Two-point Padé expansions for a family of analytic functions*, J. Comput. Appl. Math. **9** (1983), 105–123.
3. W. B. Jones, W. J. Thron, and H. Waadeland, *A strong Stieltjes moment problem*, Trans. Amer. Math. Soc. **261** (1980), 503–528.

4. O. Njåstad and W. J. Thron, *The theory of sequences of L -polynomials*, Padé Approximants and Continued Fractions (H. Waadeland and H. Wallin, eds.), Det Kongelige Norsk Videnskabers Selskab, vol. 1, Universitetsforlaget, Trondheim, 1983, pp. 54–91.
5. A. Sri Ranga, *On a recurrence formula associated with strong distributions*, SIAM J. Math. Anal. **21** (1990), 1335–1348.
6. A. Sri Ranga and J. H. McCabe, *On the extensions of some classical distributions*, Proc. Edinburgh Math. Soc. **34** (1991), 61–67.
7. A. Sri Ranga, *The strong c -symmetric distribution*, J. Austral. Math. Soc. Ser. A **53** (1992), 61–65.
8. ———, *Another quadrature rule of highest algebraic degree of precision*, Numer. Math. **68** (1994), 283–294.
9. A. Sri Ranga and E. X. L. de Andrade, *A weight function that appears in the limit and certain associated polynomials* (submitted).
10. A. Sri Ranga, E. X. L. de Andrade, and J. H. McCabe, *Some consequences of symmetry in strong distributions*, J. Math. Anal. Appl. (to appear).
11. A. Sri Ranga and C. F. Bracciali, *A continued fraction associated with a special Stieltjes function*, Comm. Anal. Theory Continued Fractions **3** (1994), 60–64.
12. A. Sri Ranga and J. H. McCabe, *On pairwise related strong Stieltjes distributions* (submitted).
13. G. Szegő, *Orthogonal polynomials*, 4th ed., Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, RI, 1975.

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