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THE H-SPACE SQUARING MAP ON $\Omega^3 S^{4n+1}$ FACTORS THROUGH THE DOUBLE SUSPENSION

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ABSTRACT. We compute the first EHP spectral sequence differential followed by the double suspension. We show that $2\pi_*(S^{4n+1}) \subset \operatorname{Im}(E^2)$, which refines the exponent for $\pi_*(S^{2n+1})$ of James and Selick. The proof follows an odd primary program of Gray and Harper, and uses Barratt's theory of unsuspended Hopf invariants and Boardman and Steer's geometric Hopf invariants.

1. INTRODUCTION

We work in the category of 2-local spaces. Let H_j be the left lexicographical James-Hopf invariant (see [5, Def. 3.10] or §2). We have James's EHP fibration [6]

$$\Omega^2 S^{2q+1} \xrightarrow{P} S^q \xrightarrow{E} \Omega S^{q+1} \xrightarrow{H_2} \Omega S^{2q+1}. \tag{1}$$

James [12] showed that $4\pi_*(S^{2q+1}) \subset \operatorname{Im}(E^2)$, giving an exponent of 2^{2q} for $\pi_*(S^{2q+1})$, which Selick [15] improved to $2^{2q-[q/2]}$. F. Cohen [6, §6] reformulated Selick's proof as a compression of the H-space squaring map on $\Omega^4 S^{4n+1}$ through some map $g\colon \Omega^2 S^{4n-1} \longrightarrow \Omega^4 S^{4n+1}$. We give a stronger compression result, which was conjectured by Gray and Mahowald.

Theorem 1.1. *The following diagrams commute up to homotopy.*

$$\begin{tikzcd}
\Omega^3 S^{4n+1} \arrow[r, "\Omega P"] \arrow[rrd, "2"'] & \Omega S^{2n} \arrow[r, "H_2"] & \Omega S^{4n-1} \arrow[d, "-\Omega E^2"] \\
 & & \Omega^3 S^{4n+1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega^3 S^{4n-1} \arrow[r, "H_2 \circ \Omega P"] \arrow[rd, "1+\Omega^3(-\iota)"'] & \Omega S^{4n-3} \arrow[d, "\Omega E^2"] \\
 & \Omega^3 S^{4n-1}
\end{tikzcd}$$

Thus $2\pi_*(S^{4n+1}) \subset \operatorname{Im}(E^2)$, which is suggested by Selick's exponent theorem. With Barratt, Cohen, Gray and Mahowald [3], we gave simple proofs of weaker compression theorems, and deduced that the E_2 term of the EHP spectral

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sequence is a $\mathbb{Z}/2$ module, and that 4 is the order of the identity map on $\Omega^2 W(n)$.

The proof follows the odd-primary work of Gray [9, 10] and Harper [11]. We compute the H-deviation of a map $\beta\colon \Omega S^q \longrightarrow \Omega^3 S^{2q+1}$ determined by the sequence

$$\Omega S^q \xrightarrow{\Omega E} \Omega^2 S^{q+1} \xrightarrow{\Omega H_2} \Omega^2 S^{2q+1} \xrightarrow{1+\Omega^2(-1)^q} \Omega^2 S^{2q+1} \tag{2}$$

by identifying (Theorem 2.3) the delooping of the second composite as a cup product. We dualize (Theorem 3.1) a theorem of Barratt and Toda [17, Prop. 2.6], which involves the Hopf invariant of a Toda bracket. By Boardman and Steer's Cartan formula recognition principal [5, Thm. 3.15] (Theorem 3.2), β is determined by its H-deviation to be $(-1)^{q-1}\Omega E^2 \circ H_2\colon \Omega S^q \longrightarrow \Omega^3 S^{2q+1}$.

The proof of Theorem 2.3 uses techniques from Barratt's unpublished theory of unsuspended Hopf invariants. Essentially, we prove an unstable symmetry formula for H_2. Furthermore, Barratt's technique of analyzing the Hilton Hopf expansions of commutativity $(x+y)f = (y+x)f$ and associativity $(x+(y+z))f = ((x+y)+z)f$ proves Theorem 2.3 up to James filtration 4, giving crucial evidence for the result.

Our work also owes a heavy debt to Boardman and Steer's [5] work on suspended Hopf invariants. Theorem 3.1 is based on Boardman and Steer's proof of the Cartan formula for their geometric Hopf invariant [5, Thm. 5.6, picture p. 201].

In §4 we prove two formulas about the Barratt-Ganea-Toda relative Hopf invariant [2, 7, 16, 14], which can be used to give a unified proof of both our theorem and Harper's. In §5 we attempt to give some context for our work, sketching the unified proof and discussing results related to Theorems 2.3 and 3.1.

I want to thank Michael Barratt, Brayton Gray and Mark Mahowald for many stimulating conversations leading to the proof. I would also like to thank Fred Cohen, Mike Hopkins, Jeff Smith, George Whitehead and Bruce Williams for various conversations about elementary unstable homotopy theory. I want to thank John Harper for carefully reading the manuscript and suggesting some useful changes, and pointing out Zabrodsky's version [19, Lem. 3.2.1& Prop. 3.2.2] of Theorem 3.1.

2. James Hopf invariants

We now recall from Boardman and Steer [5] the definition of smash products, cup products, James Hopf invariants and Whitehead products. See also Whitehead's book [18] for its attention to point-set topology, and for the James splitting, which is not treated in [5] or in any of James's papers.

Suspension will mean smashing on the right with $S^1 = I/\{0, 1\}$, so $\Sigma X = X \wedge S^1$. We define $S^n = (S^1)^{[n]} = I^n/\partial(I^n)$. By associativity of the smash product, $\Sigma^n X = X \wedge S^n$. For spaces A and B, the *shuffle* permutations

$$\text{shuffle}\colon \Sigma^{n+m}(A \wedge B) = A \wedge B \wedge S^n \wedge S^m \xrightarrow{1_A \wedge T \wedge 1_{S^m}} \Sigma^n A \wedge \Sigma^m B$$

are frequently used and suppressed from the notation. Given maps $\phi\colon \Sigma^n A \longrightarrow X$ and $\gamma\colon \Sigma^m A \longrightarrow Y$, the *cup product* $\phi \cdot \gamma$ is defined to be the composite

$$\phi \cdot \gamma\colon \Sigma^{n+m} A \xrightarrow{\Sigma^{n+m}\Delta} \Sigma^n A \wedge \Sigma^m A \xrightarrow{\phi\wedge\gamma} X \wedge Y.$$

We define adjoints between suspensions and loop spaces following Zabrodsky [19, §0.3]. Given pointed spaces A, X and K, and pointed maps $f\colon A \wedge K \longrightarrow X$ and $g\colon A \longrightarrow X^K$, we denote their adjoints by $f^\vee\colon A \longrightarrow X^K$ and $g^\wedge\colon A \wedge K \longrightarrow X$. Given a map $f\colon A \longrightarrow \Omega^2 X = (\Omega X)^{S^1} \cong X^{S^2}$, it will be clear from context which of the adjoints $f^\wedge\colon \Sigma A \longrightarrow \Omega X$ and $f^\wedge\colon \Sigma^2 A \longrightarrow X$ we mean.

Let X be a connected CW complex with basepoint $* \in X$. The James construction $J(X) = \coprod_{k=1}^\infty X^k/\sim$ is naturally homotopy equivalent [5, Thm. 3.3] to $\Omega\Sigma X$. Let $\iota_k\colon X^k \longrightarrow \Omega\Sigma X$ be the adjoint of the "sum" $\iota_k^\wedge = \sum_{i=1}^k \Sigma\pi_i\colon \Sigma X^k \longrightarrow \Sigma X$. The *James-Hopf invariant* $H_j\colon \Omega\Sigma X \longrightarrow \Omega\Sigma X^{[j]}$ is defined so that [5, Lem. 3.11]

$$H_j^\wedge \circ \Sigma\iota_k \simeq \sum_{1\le\sigma_1<\cdots<\sigma_j\le k} \Sigma(\pi_{\sigma_1}\cdot\pi_{\sigma_2}\cdot\cdots\cdot\pi_{\sigma_j})\colon \Sigma(X^k) \longrightarrow \Sigma X^{[j]}, \tag{3}$$

where the sum is ordered left lexicographically. The map H_j is uniquely determined by (3). This follows from the James splitting [18, Thm. VII(2.10)], the homotopy equivalence $\Sigma\Omega\Sigma X \xrightarrow{\simeq} \vee_{i=1}^\infty \Sigma X^{[k]}$ obtained by adding up the Hopf invariants.

Given $f\colon \Sigma A \longrightarrow M$ and $g\colon \Sigma B \longrightarrow M$, the *Whitehead product* [5, Def. 4.2] $[f, g]\colon \Sigma A \wedge B \longrightarrow M$ is defined to be the unique homotopy class such that

$$\begin{aligned}(\Sigma\pi_{12})^*[f, g] &= (f\circ\pi_1, g\circ\pi_2)\\ &= -f\circ\pi_1 - g\circ\pi_2 + f\circ\pi_1 + g\circ\pi_2 \in [\Sigma(A\times B), M].\end{aligned}$$

For any maps $f, g\colon \Sigma A \longrightarrow \Sigma X$, we have $(f, g) = [f, g]\circ\Sigma(\Delta) \in [\Sigma A, \Sigma X]$. Given spaces and maps $\alpha\colon A \longrightarrow X$, $\beta\colon A \longrightarrow Y$, $f\colon \Sigma X \longrightarrow M$ and $g\colon \Sigma Y \longrightarrow M$, it follows from naturality that $(f\circ\Sigma\alpha, g\circ\Sigma\beta) = [f, g]\circ\Sigma(\alpha\cdot\beta) \in [\Sigma A, \Sigma X]$. Recall the algebraic identities $gf = fg(g, f)$ and $(gh, f) = (g, f)((g, f), h)(h, f)$, for elements f, g, h of a group. Therefore, for any maps $f, g, h\colon A \longrightarrow X$,

$$\Sigma g + \Sigma f = \Sigma f + \Sigma g + [\iota, \iota]\circ\Sigma(g\cdot f) \in [\Sigma A, \Sigma X]. \tag{4}$$

$$\begin{aligned}\Sigma g + \Sigma h + \Sigma f =& \Sigma f + \Sigma g + \Sigma h + [\iota, \iota]\circ\Sigma(g\cdot f)\\ &+ [[\iota, \iota], \iota]\circ\Sigma(g\cdot f\cdot h) + [\iota, \iota]\circ\Sigma(h\cdot f) \in [\Sigma A, \Sigma X].\end{aligned} \tag{5}$$

The symmetric group Σ_k acts on $X^{[k]}$, for any space X, by

$$\sigma(x_1\wedge\cdots\wedge x_k) = x_{\sigma^{-1}(1)}\wedge\cdots\wedge x_{\sigma^{-1}(k)}.$$

Note that the $\sigma(i)^{\text{th}}$ coordinate of $\sigma(x_1\wedge\cdots\wedge x_k)$ is x_i. Note that if $X = S^q$, then for any $\sigma \in \Sigma_k$, the permutation $\sigma\colon X^{[k]} \longrightarrow X^{[k]}$ is homotopic to the degree $(-1)^{k\,\mathrm{sgn}(\sigma)}$ map, under the canonical homeomorphism $S^{kq} \cong X^{[k]}$.

Now we discuss Barratt's theory of unsuspended Hopf invariants. For a preliminary account of this theory see [4], which unfortunately contains serious mistakes. In particular Baues claims [4, Prop. II(2.15) & Prop. III(5.3)] a Cartan formula for H_j which is false and incompatible with Theorem 2.3 below. We begin with

Lemma 2.1. *Let X be a space and $f, g\colon X \longrightarrow S^{2q-1}$ be two maps such that $E^2 f = E^2 g \in [\Sigma^2 X, S^{2q+1}]$. Then the composites $[\iota, \iota]\circ f, [\iota, \iota]\circ g\colon X \longrightarrow S^q$ are homotopic.*

Proof. The Whitehead product $[\iota, \iota]\colon S^{2q-1} \longrightarrow S^q$ suspends to zero. By the EHP fibration (1), $[\iota, \iota]$ factors through P by some map $\kappa\colon S^{2q-1} \longrightarrow \Omega^2 S^{2q+1}$. By [18, Thm. XII(2.4)], $\kappa \simeq (-1)^{q-1}E^2$, but we do not need this. Then the composites $\kappa\circ f, \kappa \circ g\colon X \longrightarrow \Omega^2 S^{2q+1}$ are homotopic. $\square$

By the Jacobi identity (cf. [6, Cor. 1.3]), the triple Whitehead product $[[\iota, \iota], \iota] \in \pi_{3q+1}(S^{q+1})$ is zero. Hence, by (4), (5) and Lemma 2.1, we have

Lemma 2.2. *Given a space A, maps $f_1, \dots, f_l\colon A \longrightarrow S^q$, and $\sigma \in \Sigma_l$,*

$$\sum_{i=1}^{l} \Sigma f_i = \sum_{j=1}^{l} \Sigma f_{\sigma(j)} + [\iota, \iota]\circ \sum_{i<\sigma(j),\ \sigma^{-1}(i)>j} \Sigma(f_i \cdot f_{\sigma(j)}) \in [\Sigma A, S^{q+1}],$$

where the sum of commutators is ordered arbitrarily.

For spaces X and Y, let $\sigma\colon \Sigma\Omega X \longrightarrow X$ be the evaluation map. We define $\otimes\colon \Omega\Sigma X \wedge \Omega\Sigma Y \longrightarrow \Omega\Sigma(X\wedge Y)$ to be the adjoint of the composite

$$\Sigma(\Omega\Sigma X \wedge \Omega\Sigma Y) \xrightarrow{\sigma\wedge \mathrm{id}} \Sigma(X \wedge \Omega\Sigma Y) \xrightarrow{\mathrm{id}\wedge\sigma} \Sigma(X\wedge Y).$$

Let $\bar{k} = \{1, \dots, k\}$, and let $\prec$ and $\prec_r$ denote the left and right lexicographical order on $\bar{k}^2$. Then the composite $\Sigma X^k \xrightarrow{\Sigma(\iota_k)} \Sigma\Omega\Sigma X \xrightarrow{\Sigma\Delta} \Sigma\Omega\Sigma X \wedge \Omega\Sigma X \xrightarrow{\hat{\otimes}} \Sigma X \wedge X$ is the sum $\sum^{\prec}_{(i,j)\in\bar{k}^2} \Sigma(\pi_i\cdot\pi_j)$, meaning we sum in the left lexicographical order. Note that we can drop the terms $i=j$ if X is a suspension. We now prove

Theorem 2.3. *The following diagram is homotopy commutative.*

$$\begin{array}{ccc} \Omega S^{q+1} & \xrightarrow{H_2} & \Omega S^{2q+1} \\ {\scriptstyle\Delta}\big\downarrow & & \big\downarrow{\scriptstyle 1+\Omega(-1)^q} \\ (\Omega S^{q+1})^{[2]} & \xrightarrow{\otimes} & \Omega S^{2q+1} \end{array}$$

Proof. For convenience, let $X = S^q$. Then we have $(12)\simeq(-1)^q\colon X\wedge X \longrightarrow X \wedge X$. We thus need to prove that $\hat{H_2} + \Sigma(12)\circ \hat{H_2} \simeq \hat{\otimes}\circ\Sigma(\Delta)\colon \Sigma\Omega\Sigma X \longrightarrow \Sigma X^{[2]}$, or that

$$(6)\qquad (\hat{H_2} + \Sigma(12)\circ\hat{H_2})\circ\Sigma\iota_k \simeq \sum^{\prec}_{\{(i,j)\in\bar{k}^2\,|\,i\neq j\}} \Sigma(\pi_i\cdot\pi_j)\colon \Sigma X^k \longrightarrow \Sigma X\wedge X.$$

Equation (3) now reads $H_2^\frown \circ \Sigma\iota_k = \sum^{\prec}_{\{(i,j)\in\bar{k}^2 \mid i<j\}} \Sigma(\pi_i \cdot \pi_j)$, and we have

$$\Sigma(12)\circ H_2^\frown \circ \Sigma\iota_k = \sum^{\prec}_{\{(\alpha,\beta)\in\bar{k}^2 \mid \alpha<\beta\}} \Sigma(\pi_\beta \cdot \pi_\alpha) = \sum^{\prec_r}_{\{(i,j)\in\bar{k}^2 \mid i>j\}} \Sigma(\pi_i \cdot \pi_j),$$

since $(12)\circ(\pi_i \cdot \pi_j) = \pi_j \cdot \pi_i : X^k \longrightarrow X^{[2]}$. Now we use Lemma 2.2 to rewrite this sum in left lexicographical order. Let i, j, α, β be distinct indices with $i > j$ and $\alpha > \beta$. Then $(\alpha, \beta) \prec_r (i, j)$ and $(i, j) \prec (\alpha, \beta)$ iff $\beta < j < i < \alpha$. Hence

$$\begin{aligned}\Sigma(12)\circ H_2^\frown \circ \Sigma\iota_k &= \sum^{\prec}_{\{(i,j)\in\bar{k}^2 \mid i>j\}} \Sigma(\pi_i \cdot \pi_j) + [\iota, \iota]\circ \sum_{\beta<j<i<\alpha} \Sigma(\pi_\alpha \cdot \pi_\beta \cdot \pi_i \cdot \pi_j)\\ &= \sum^{\prec}_{i>j} \Sigma(\pi_i \cdot \pi_j) + [\iota, \iota]\circ\Sigma(124)\circ H_4^\frown \circ \Sigma\iota_k,\end{aligned}$$

since we can order the commutators left lexicographically. By Lemma 2.2 we have

$$\sum^{\prec}_{i<j} \Sigma(\pi_i \cdot \pi_j) + \sum^{\prec}_{i>j} \Sigma(\pi_i \cdot \pi_j) = \sum^{\prec}_{i\neq j} \Sigma(\pi_i \cdot \pi_j) + [\iota, \iota]\circ\Sigma(1423)\circ H_4^\frown \circ \Sigma\iota_k,$$

picking up a commutator $[\iota, \iota]\circ\Sigma(\pi_\alpha \cdot \pi_\beta \cdot \pi_i \cdot \pi_j) = [\iota, \iota]\circ(1423)\circ \Sigma(\pi_j \cdot \pi_i \cdot \pi_\alpha \cdot \pi_\beta)$ for each $\alpha < \beta$ and $i > j$ with $(i, j) \prec (\alpha, \beta)$, i.e. $i < \alpha$. Furthermore $(124) \simeq \mathrm{id}$ and $(1423) \simeq (-1)^q$. Recall [6, Cor. 1.3] that $[\iota, \iota]\colon \Sigma X^{[4]} \longrightarrow \Sigma(X\wedge X) = S^{2q+1}$ has order two. Thus $[\iota, \iota]\circ\Sigma(1423) \simeq [\iota, \iota]\circ(-1)^q = (-1)^q[\iota, \iota] \simeq [\iota, \iota]$. Hence

$$(H_2^\frown + \Sigma(12)\circ H_2^\frown)\circ\Sigma\iota_k = \sum^{\prec}_{\{(i,j)\in\bar{k}^2 \mid i\neq j\}} \Sigma(\pi_i \cdot \pi_j) + [\iota, \iota]\circ H_4^\frown \circ (2\iota)\circ\Sigma\iota_k.$$

By Lemma 2.1, $[\iota, \iota]\circ H_4^\frown \circ(2\iota) = [\iota, \iota]\circ(2\iota)\circ H_4^\frown$, since $H_4^\frown \circ (2\iota)$ and $(2\iota)\circ H_4^\frown$ are equal after a suspension. But

$$[\iota, \iota]\circ(2\iota) = 2[\iota, \iota] = 0 \in [\Sigma X^{[4]}, \Sigma(X\wedge X)]. \quad \square$$

3. H-DEVIATIONS, CUP PRODUCTS AND THE CARTAN FORMULA

For spaces F and X and a map $f\colon \Omega F \longrightarrow \Omega^2 X$, with adjoint $f^\frown\colon \Sigma^2\Omega F \longrightarrow X$, we define the *H-deviation* $D(f)\colon (\Sigma\Omega F)^{[2]} \longrightarrow X$ to be the homotopy class so that

(7)
$$f^\frown \circ \Sigma^2\mu = f^\frown \circ \Sigma^2\pi_1 + D(f)\circ(\Sigma\pi_1 \cdot \Sigma\pi_2) + f^\frown \circ \Sigma^2\pi_2 \in [\Sigma^2(\Omega F\times\Omega F), X],$$

where $\Sigma\pi_1 \cdot \Sigma\pi_2$ differs from $\Sigma^2\pi_{12}\colon \Sigma^2(\Omega F\times\Omega F) \longrightarrow \Sigma^2(\Omega F\wedge\Omega F)$ by a shuffle. The Barratt splitting [1] $\Sigma(\Omega F\times\Omega F) \xrightarrow{x\Sigma\pi_1 + y\Sigma\pi_{12} + z\Sigma\pi_2} \Sigma\Omega F \vee \Sigma(\Omega F\wedge\Omega F)\vee\Sigma\Omega F$ shows that $D(f)$ is uniquely defined by (7). For any space A, let $f_*\colon [\Sigma A, F] \longrightarrow [\Sigma^2 A, X]$ denote the natural transformation sending the map $\alpha\colon \Sigma A \longrightarrow F$ to the composite $f_*(\alpha)\colon \Sigma^2 A \xrightarrow{\Sigma^2\alpha^\vee} \Sigma^2\Omega F \xrightarrow{f^\frown} X$.

Then for $\xi_1, \xi_2\colon \Sigma A \longrightarrow F$, we have

$$f_*(\xi_1+\xi_2) = f_*(\xi_1) + D(f)\circ(\Sigma\xi_1^\vee \cdot \Sigma\xi_2^\vee) + f_*(\xi_2) \in [\Sigma^2 A, X]. \tag{8}$$

3.1. **The dual Barratt-Toda formula.** The Hilton-Eckmann dual of the Hopf invariant of a Toda bracket is the H-deviation of a *colifting* (cf. [3, Lem. 2.1]) of a sequence such as (2). We will not use colifting explicitly, or prove that our map β below is induced by some nullhomotopy of the composite $\Omega E \longrightarrow \Omega B \longrightarrow \Omega X$, although this was our motivation. Now we use Boardman and Steer's geometric proof of the Cartan formula [5, Thm. 5.6] to prove a dual Barratt-Toda formula.

Theorem 3.1. *Assume we have the homotopy commutative diagram*

$$\begin{tikzcd} \Omega B \arrow[r, "\partial"] & F \arrow[r, "\pi"] & E \arrow[r, "p"] \arrow[d, "\Delta"'] & B \arrow[d, "f"] \\ & & E\wedge E \arrow[r, "\alpha"] & X \end{tikzcd}$$

where F is the homotopy fiber of p, with boundary map $\Omega B \xrightarrow{\partial} F$. Then there exists a map $\beta\colon \Omega F \longrightarrow \Omega^2 X$ such that the following diagrams are homotopy commutative.

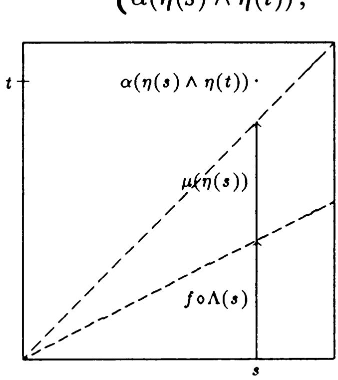
$$\begin{tikzcd} \Omega^2 B \arrow[rr, "\Omega\partial"] \arrow[dr, "\Omega^2 f"'] & & \Omega F \arrow[dl, "\beta"] \\ & \Omega^2 X & \end{tikzcd} \qquad \begin{tikzcd} (\Sigma\Omega F)^{[2]} \arrow[r, "(\Sigma\Omega\pi)^{[2]}"] \arrow[d, "D(\beta)"'] & (\Sigma\Omega E)^{[2]} \arrow[d, "\sigma^{[2]}"] \\ X & E^{[2]} \arrow[l, "\alpha"'] \end{tikzcd}$$

Proof. Recall that $F = \{(e,\lambda) : \lambda(1) = p(e)\} \subset E \times PB$. Write elements of ΩF as $(\eta, \Lambda)\colon S^1 \longrightarrow F$, so that $\forall s \in S^1$, $\Lambda(s)(1) = p(\eta(s))$. Let $\mu\colon E \longrightarrow X^I$ be a homotopy of the diagram, with $\mu(0) = f\circ p$, $\mu(1) = \alpha\circ\Delta$. We define the map $\beta\colon \Omega F \longrightarrow \Omega^2 X$ by (see picture (9))

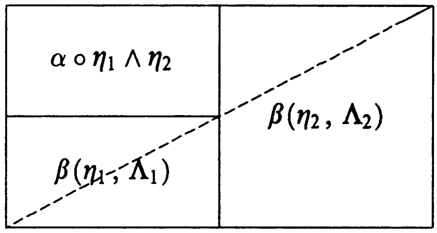
$$\Sigma^2\Omega F \xrightarrow{\hat{\beta}} X$$

$$\hat{\beta}((\eta,\Lambda)\wedge s\wedge t) = \begin{cases} f(\Lambda(s)(\frac{2t}{s})), & 0 \le t \le \frac{s}{2}, \\ \mu(\eta(s))(\frac{2t}{s}-1), & \frac{s}{2} \le t \le s, \\ \alpha(\eta(s)\wedge\eta(t)), & s \le t \le 1. \end{cases}$$

(9)



Then, following Boardman and Steer [5], we compute the H-deviation of β. For $(\eta_1, \Lambda_1), (\eta_2, \Lambda_2) \in \Omega F$, $\beta((\eta_1, \Lambda_1) + (\eta_2, \Lambda_2)) \in \Omega^2 X$ is represented by the picture



where the solid lines are mapped to $* \in X$. Thus (cf. [5, Lem. 5.7]) the composite $\Sigma^2(\Omega F \times \Omega F) \xrightarrow{\Sigma^2\mu} \Sigma^2\Omega F \xrightarrow{\beta^\frown} X$ is homotopic to the sum of three maps; we have

$$\beta^\frown \circ \Sigma^2\mu \simeq \beta^\frown \circ \Sigma^2\pi_1 + \alpha \circ \sigma \wedge \sigma \circ \Sigma\pi_1 \cdot \Sigma\pi_2 + \beta^\frown \circ \Sigma^2\pi_2 \colon \Sigma^2(\Omega F \times \Omega F) \longrightarrow X.$$

By (7) we have calculated $D(\beta)$, so the right-hand square homotopy commutes.

For any loop $\Lambda \in \Omega^2 B$, $\partial(\Lambda) = (*, \Lambda) \in F$ and $\beta(*, \Lambda) \in \Omega^2 X$ is a dilation of $f \circ \Lambda$ to the lower triangle of the square (9). Thus the triangle homotopy commutes. □

3.2. **Boardman and Steer's Cartan formula recognition principle.** The natural transformation $\lambda_2 \colon [\Sigma A, \Sigma Y] \longrightarrow [\Sigma^2 A, (\Sigma Y)^{[2]}]$ satisfies a *Cartan formula* [5, Thm. 2.2], where λ_2 is induced by the composite, which we will also call λ_2,

$$\lambda_2 \colon \Omega\Sigma Y \xrightarrow{H_2} \Omega\Sigma Y^{[2]} \xrightarrow{\Omega(E)} \Omega^2(\Sigma^2 Y^{[2]}) \xrightarrow{\Omega^2(\text{shuffle})} \Omega^2(\Sigma Y)^{[2]};$$
$$\lambda_2(\xi_1 + \xi_2) = \lambda_2(\xi_1) + \xi_1 \cdot \xi_2 + \lambda_2(\xi_2) \colon \Sigma^2 A \longrightarrow (\Sigma Y)^{[2]} \qquad \text{for } \xi_i \colon \Sigma A \longrightarrow \Sigma Y.$$

λ_2 is adjoint to $\Sigma H_2^\frown$, and similar to equation (3), λ_2 is characterized by

$$(10) \qquad \lambda_2^\frown \circ \Sigma^2\iota_k \simeq \sum_{1 \le i < j \le k} \Sigma\pi_i \cdot \Sigma\pi_j \colon \Sigma^2(Y^k) \longrightarrow (\Sigma Y)^{[2]}.$$

Theorem 3.2. *Given $\beta \colon \Omega X \longrightarrow \Omega^3\Sigma X^{[2]}$ for a suspension $X = \Sigma Y$ of a CW complex Y, with $Y \xrightarrow{E} \Omega X \xrightarrow{\beta} \Omega^3\Sigma X^{[2]}$ nullhomotopic, assume that the H-deviation of β is $D(\beta) \colon (\Sigma\Omega X)^{[2]} \xrightarrow{\sigma\wedge\sigma} X^{[2]} \xrightarrow{E} \Omega\Sigma X^{[2]}$. Then β is homotopic to the composite*

$$\Omega X \xrightarrow{H_2} \Omega\Sigma Y^{[2]} \xrightarrow{\Omega E} \Omega^2\Sigma^2 Y^{[2]} \xrightarrow{\Omega^2(\text{shuffle})} \Omega^2 X^{[2]} \xrightarrow{\Omega^2 E} \Omega^3\Sigma X^{[2]}.$$

Furthermore, if $X = S^q$, then $\beta \simeq (-1)^{q-1}\Omega E^2 \circ H_2 \colon \Omega S^q \longrightarrow \Omega^3 S^{2q+1}$.

Proof. The composite above is $\Omega^2 E \circ \lambda_2 \colon \Omega\Sigma Y \longrightarrow \Omega\Sigma(\Sigma Y)^{[2]}$. By assumption and (8), for a space A and maps $\xi_1, \xi_2 \colon \Sigma A \longrightarrow X$, we have

$$(11) \qquad \beta_*(\xi_1 + \xi_2) = \beta_*(\xi_1) + E \circ (\xi_1 \cdot \xi_2) + \beta_*(\xi_2) \in [\Sigma^2 A, \Omega\Sigma X^{[2]}].$$

Following the proof of [5, Thm. 3.15], we compute

$$\beta^\frown \circ \Sigma^2 \iota_k = (\beta \circ \iota_k)^\frown = \beta_*(\Sigma\pi_1 + \cdots + \Sigma\pi_k) = E \circ \lambda_2^\frown \circ \Sigma^2 \iota_k \in [\Sigma^2 Y^k, \Omega\Sigma X^{[2]}].$$

Hence by the characterization (10), $\beta^\frown \simeq E \circ \lambda_2^\frown$, which verifies the first part. If $X = S^q$, then shuffle $\simeq (-1)^{q-1}$ and

$$\beta^\frown = E \circ (-1)^{q-1} \circ \Sigma H_2^\frown = (-1)^{q-1}(E \circ \Sigma H_2^\frown). \quad \square$$

Proof of Theorem 1.1. Theorem 3.1, fibration (1) and Theorem 2.3 yield a map $\beta\colon \Omega S^q \longrightarrow \Omega^3 S^{2q+1}$ such that the composite $\beta \circ \Omega P$ is $1 + \Omega^3(-1)^q$. Theorem 3.2 identifies β to be $(-1)^{q-1}\Omega E^2 \circ H_2$, and the following diagram is homotopy commutative:

$$\begin{tikzcd}
\Omega^3 S^{2q+1} \arrow[r, "\Omega P"] \arrow[dr, "1+\Omega^3(-1)^q"'] & \Omega S^q \arrow[r, "H_2"] & \Omega S^{2q-1} \arrow[dl, "(-1)^{q-1}\Omega E^2"] \\
& \Omega^3 S^{2q+1} &
\end{tikzcd} \quad \square$$

4. The relative Hopf invariant

For a mapping cone $M = Y \cup CA$ of an NDR pair (Y, A), let $\theta\colon M \longrightarrow Y/A \vee \Sigma A$ be the *coaction map* (cf. [7, 8]), obtained by pinching out the subspace A. Let $\theta_1\colon M \longrightarrow Y/A$ and $\theta_2\colon M \longrightarrow \Sigma A$ be the composites of θ with the projections onto the two factors. We define the *relative Hopf invariant* $\mathfrak{h}\colon \Omega M \longrightarrow \Omega^2(Y/A \wedge \Sigma A)$ by

$$\mathfrak{h}^\frown(\omega \wedge s \wedge t) = \begin{cases} \theta_1(\omega(s)) \wedge \theta_2(\omega(t)), & 0 \le s \le t \le 1, \\ *, & \text{otherwise}, \end{cases} \qquad \text{for } \omega \in \Omega M.$$

In the absolute case $M = \Sigma A$, $\mathfrak{h}\colon \Omega\Sigma A \longrightarrow \Omega^2(\Sigma A \wedge \Sigma A)$ is Boardman and Steer's geometric Hopf invariant [5, Def. 5.4]. Again following [5, Thm. 5.6], we have

Theorem 4.1. *Let $\Omega B \xrightarrow{\partial} F \xrightarrow{\pi} E$ be the principal fibration induced from a map $p\colon E \longrightarrow B$. Assume that E has an NDR subspace M such that $p(M) = *$, and let $\mathfrak{e}\colon M \longrightarrow F$ be the lift of $M \rightarrowtail E$ given by the trivial nullhomotopy. Let $M = Y \cup CA$ be the mapping cone of the NDR pair (Y, A), and let $\bar\Delta\colon E/M \longrightarrow E/CA \wedge E/Y$ be the relative diagonal map. Suppose the following diagram is homotopy commutative.*

$$\begin{tikzcd}
E/M \arrow[r, "\bar p"] \arrow[d, "\bar\Delta"'] & B \arrow[d, "f"] \\
E/CA \wedge E/Y \arrow[r, "\alpha"] & X
\end{tikzcd} \tag{12}$$

Let $\iota\colon Y/A \wedge \Sigma A \rightarrowtail E/CA \wedge E/Y$ be the inclusion arising from $M \rightarrowtail E$. Then there exists a map $\beta\colon \Omega F \longrightarrow \Omega^2 X$ making the following diagram homotopy *commutative.*

$$\begin{tikzcd}
\Omega^2 B \arrow[rr, "\Omega\partial"] \arrow[d, "\Omega^2 f"'] & & \Omega F \arrow[dll, "\beta"'] & & \Omega M \arrow[ll, "\Omega\epsilon"'] \arrow[d, "\mathfrak{h}"] \\
\Omega^2 X & & \Omega^2(E/CA \wedge E/Y) \arrow[ll, "\Omega^2\alpha"'] & & \Omega^2(Y/A \wedge \Sigma A) \arrow[ll, "\Omega^2\iota"']
\end{tikzcd}$$

If X is a loop space and there is a homotopy retraction $\Sigma E \longrightarrow \Sigma M$ of $M \rightarrowtail E$, then it suffices for the next diagram to be homotopy commutative instead of (12).

$$\begin{tikzcd}
E \arrow[r, "p"] \arrow[d, "\Delta"'] & B \arrow[d, "f"] \\
E \wedge E/Y \arrow[r, "\alpha"] & X
\end{tikzcd} \tag{13}$$

Proof. Let $\mu\colon E/M \longrightarrow X^I$ be a homotopy of the diagram (12), so that $\mu(0) = f \cdot \bar{p}$ and $\mu(1) = \alpha \cdot \bar{\Delta}$. Let $\pi\colon E \longrightarrow E/M$, $\pi_1\colon E \longrightarrow E/CA$ and $\pi_2\colon E \longrightarrow E/Y$ be the collapse maps. Then β is defined similarly to (9); $\beta^\frown\colon \Sigma^2\Omega F \longrightarrow X$ is defined by

$$X \ni \beta^\frown((\eta, \Lambda) \wedge s \wedge t) = \begin{cases} f(\Lambda(s)(\frac{2t}{s})), & 0 \le t \le \frac{s}{2}, \\ \mu(\pi(\eta(s))(\frac{2t}{s} - 1), & \frac{s}{2} \le t \le s, \\ \alpha(\pi_1(\eta(s)) \wedge \pi_2(\eta(t))), & s \le t \le 1, \end{cases} \qquad \text{for } (\eta, \Lambda) \in \Omega F.$$

The homotopy commutativity of the left triangle follows immediately, as in the proof of Theorem 3.1, while the right five-sided figure commutes strictly.

Assuming the stronger hypotheses and adjointing, we see that we have an injection $[E/M, X] \rightarrowtail [E, X]$. Then diagram (13) implies the stronger (12). □

Given a map $f\colon A \longrightarrow Y$ with mapping cone $M = Y \cup_f CA$, let $\theta\colon M \longrightarrow M \vee \Sigma A$ be the coaction map obtained by pinching out the NDR subspace $A \times \frac{1}{2} \subset M$. We similarly define the relative Hopf invariant $\mathfrak{h}\colon \Omega M \longrightarrow \Omega^2(M \wedge \Sigma A)$. Given any spaces X and Y, let $X \flat Y$ be the fiber of the map $X \vee Y \rightarrowtail X \times Y$ (cf. [7]). Let $\rho\colon X\flat Y \longrightarrow \Omega(X \wedge Y)$ be the natural map from the fiber to the loops of the cofiber. There is a splitting $\Omega(X\flat Y) \times \Omega(Y) \times \Omega(X) \xrightarrow{\simeq} \Omega(X \vee Y)$, which defines a projection $\pi_\flat\colon \Omega(X \vee Y) \longrightarrow \Omega(X\flat Y)$. Then $\Omega(\iota) \circ \pi_\flat\colon \Omega(X \vee Y) \longrightarrow \Omega(X \vee Y)$ is homotopic to the map which sends ω to $\omega - \iota_1 \circ \pi_1 \circ \omega - \iota_2 \circ \pi_2 \circ \omega$. Boardman and Steer's result [5, Thm. 5.12] can easily be translated to prove the following, which relates $\mathfrak{h}$ to the Toda-Hopf invariant [16, 14] $H'\colon \Omega J_{p-1}(S^{2n}) \longrightarrow \Omega S^{2np-1}$.

Theorem 4.2. *The relative Hopf invariant $\mathfrak{h}\colon \Omega M \longrightarrow \Omega^2(M \wedge \Sigma A)$ of the mapping cone $M = Y \cup_f CA$ is homotopic to the negative of the composite*

$$\Omega M \xrightarrow{\Omega(\theta)} \Omega(M \vee \Sigma A) \xrightarrow{\pi_\flat} \Omega(M\flat\Sigma A) \xrightarrow{\Omega(\rho)} \Omega^2(M \wedge \Sigma A).$$

5. Remarks

In this section we indicate how the results of §4 can be used to give a unified proof of our result and Harper's. First we need another James-Hopf invariant formula. Let $\theta_k = (12\cdots k)$ be the cyclic permutation, which acts on $X^{[p]}$ for $k \le p$. Then

Lemma 5.1. *For any p and any connected CW complex X, the following diagram is homotopy commutative.*

$$\begin{tikzcd}
\Omega\Sigma X \arrow[r, "H_p"] \arrow[d, "\Delta"'] & \Omega\Sigma X^{[p]} \arrow[rr, "1+\Omega\theta_2+\cdots+\Omega\theta_p"] & & \Omega\Sigma X^{[p]} \arrow[d, "\Omega(E)"] \\
(\Omega\Sigma X)^{[2]} \arrow[r, "\mathrm{id}\wedge H_{p-1}"] & \Omega\Sigma X \wedge \Omega\Sigma X^{[p-1]} \arrow[r, "\otimes"] & \Omega\Sigma X^{[p]} \arrow[r, "\Omega(E)"] & \Omega^2\Sigma^2 X^{[p]}
\end{tikzcd}$$

Proof. It is enough to check the diagram restricted to each $\iota_k\colon X^k \longrightarrow \Omega\Sigma X$, or that

$$\sum_{t\in\bar{p}\ ,\ 1\le\sigma_1<\cdots<\sigma_p\le k} \Sigma^2(\pi_{\sigma_t}\cdot\pi_{\sigma_1}\cdot\cdots\cdot\pi_{\sigma_{t-1}}\cdot\pi_{\sigma_{t+1}}\cdot\cdots\cdot\pi_{\sigma_p})$$
$$= \sum_{l\in\bar{k}\ ,\ 1\le\tau_1<\cdots<\tau_{p-1}\le k\ ,\ l\notin\{\tau_1,\dots,\tau_{p-1}\}} \Sigma^2(\pi_l\cdot\pi_{\tau_1}\cdot\cdots\cdot\pi_{\tau_{p-1}})$$

in the abelian group $[\Sigma^2(X^k), \Sigma^2X^{[j]}]$. Note that $p\binom{k}{p} = k\binom{k-1}{p-1}$, and that for each term on the right-hand side, there is a unique $t\in\bar{k}$ such $\tau_{t-1} < l < \tau_t$. □

For any connected CW complex X, the James-Hopf invariant $H_k\colon J(X) \longrightarrow \Omega\Sigma X^{[k]}$ factors through a map $\overline{H_k}\colon J(X)/J_{k-1}(X) \longrightarrow \Omega\Sigma X^{[k]}$, by construction.

Lemma 5.2. *Localized at an odd prime p, the following diagram is homotopy commutative:*

$$\begin{tikzcd}
J(S^{2n}) \arrow[rr, "H_p"] \arrow[d, "\Delta"'] & & \Omega S^{2np+1} \arrow[d, "p"] \\
J(S^{2n})\wedge J(S^{2n})/J_{p-2}(S^{2n}) \arrow[r, "H_1\wedge\overline{H_{p-1}}"] & \Omega S^{2n+1}\wedge\Omega S^{2n(p-1)+1} \arrow[r, "\otimes"] & \Omega S^{2np+1}.
\end{tikzcd}$$

Proof. Since ΩS^{2np+1} is a retract of $\Omega^2 S^{2np+2}$, we can deloop Lemma 5.1. □

The James construction $J_k(X)$ of a suspension $X = \Sigma Y$ is the mapping cone of a map $f_k\colon \Sigma^{k-1}Y^{[k]} \longrightarrow J_{k-1}(X)$. Let $\theta\colon J_k(X) \longrightarrow J_k(X)\vee X^{[k]}$ be the coaction map and $\mathfrak{h}\colon \Omega J_k(X) \longrightarrow \Omega^2(J_k(X)\wedge X^{[k]})$ be the relative Hopf invariant. By shuffling a suspension coordinate of $X^{[k]} \cong \Sigma^k Y^{[k]}$, we have $\widehat{H_1}\wedge \mathrm{id}\colon J_k(X)\wedge X^{[k]} \longrightarrow X^{[k+1]}$, which one can show is a desuspension of the composite

$$J_k(X) \wedge X^{[k]} \longrightarrow J(X) \wedge J(X)/J_{k-1}(X) \xrightarrow{H_1 \wedge \overline{H_k}} \Omega\Sigma X \wedge \Omega\Sigma X^{[k]} \xrightarrow{\otimes} \Omega\Sigma X^{[k+1]}.$$

Now localize at a prime p, even or odd, and let $X = S^{2n}$. The composite of the relative Hopf invariant and $\widehat{H_1} \wedge \mathrm{id}\colon J_p(X) \wedge X^{[p-1]} \longrightarrow X^{[p]}$ gives a map $\mathfrak{h}_p\colon \Omega J_{p-1}(X) \longrightarrow \Omega^2 X^{[p]}$. We have the EHP fibration of James and Toda

$$\Omega^2\Sigma X^{[p]} \xrightarrow{\partial} J_{p-1}(X) \longrightarrow J(X) \xrightarrow{H_p} \Omega\Sigma X^{[p]}.$$

As an immediate corollary of Theorem 4.1 and the above discussion, we have

Theorem 5.3. *There exists the following p-local homotopy commutative diagram.*

$$\begin{tikzcd} \Omega^3\Sigma X^{[p]} \arrow[r, "\Omega\partial"] \arrow[dr, "p"'] & \Omega J_{p-1}(X) \arrow[r, "\mathfrak{h}_p"] & \Omega^2 X^{[p]} \arrow[dl, "\Omega^2 E"] \\ & \Omega^3\Sigma X^{[p]} & \end{tikzcd}$$

Theorem 5.3 can be used to prove half of Theorem 1.1 as well as Harper's theorem [11]. For $p = 2$, Boardman and Steer [5, Thm. 5.12] show that $\mathfrak{h}_2 = \lambda_2$. For p odd, Theorem 4.2 shows that $\mathfrak{h}_p$ is similar to Toda's original map [16].

Our original proof of Theorem 2.3 followed James's proof that $2\Omega(H_2) \simeq *$ (cf. [6, Lem. 4.1]). We applied a Cartan-like formula

$$H_2(f+g) = H_2(f) + f \smile g + H_2(g) + [\iota_2, \iota_2] \circ H_3(f) \smile g$$

for $f, g\colon \Sigma A \longrightarrow S^{q+1}$, to the Hilton-Hopf expansion

$$0 = ((-\iota) + \iota)f = (-\iota)f + f + [-\iota, \iota] \circ H_2(f)$$

for any map $f\colon \Sigma A \longrightarrow S^{q+1}$.

Zabrodsky [19, §3] proves a result similar to Theorem 3.1. He calculates the H-deviation of the map $\beta\colon \Omega A \longrightarrow \Omega^2 C$ determined by the sequence

$$\Sigma\Omega A \xrightarrow{\sigma} A \xrightarrow{g} B \xrightarrow{f} C$$

and nullhomotopies of $\Omega g\colon \Omega A \longrightarrow \Omega B$ and $fg\colon A \longrightarrow C$. Suppose that g is a cup product $A \xrightarrow{\Delta} A \wedge A \xrightarrow{\alpha} B$. Then combining Zabrodsky's Lemma 3.2.1 and Proposition 3.2.2, we have the following homotopy commutative diagram:

$$(14) \qquad \begin{tikzcd} (\Sigma\Omega A)^{[2]} \arrow[r, "\sigma\wedge\sigma"] \arrow[d, "D(\beta)"'] & A \wedge A \arrow[d, "\alpha"] \\ C & B \arrow[l, "f"'] \end{tikzcd}$$

An alternate proof of (14) can be given using the fibration $\Omega A * \Omega A \longrightarrow \Sigma\Omega A \xrightarrow{\sigma} A$ and the ΩA "projective plane"; see the discussion with references in [13, Thms. 1, 2].

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