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THE DECAY OF SUBHARMONIC FUNCTIONS OF FINITE ORDER ALONG A RAY

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ABSTRACT. A result is proved relating the growth of a subharmonic function $u(z)$ of finite lower order at least one, along a ray, to the quantity

$$B(r) = \sup\{u(z) \colon |z| < r\}.$$

This sharpens a previous result of the second author when the lower order is finite. An example is constructed to show that the result obtained is best possible.

1. INTRODUCTION

Let $u(z)$ be a subharmonic function (s.f.) in U where U denotes either the complex plane $\mathbf{C}$ or the sector $U(\theta) = \{z \colon |\arg z| < \theta\}$ where $0 < \theta \leq \pi$. If

$$B(r) = \sup_{|z|<r,\, z\in U} u(z),$$

then the lower order λ of $u(z)$ is defined by

$$\lambda = \liminf_{r\to\infty} \frac{\log B(r)}{\log r}.$$

It is well known that a s.f. u cannot decay too fast along a ray in comparison with $B(r)$. For convenience we consider the positive ray.

Theorem A. (a) *If* $0 \leq \lambda < 1$, *then*

$$u(r) \geq (\cos \pi\lambda) B(r)(1 + o(1)),$$

for a sequence $r = r_k \to \infty$.

(b) *If* $\lambda \geq 1$, *then*

$$u(r) \geq -B(r)(1 + o(1)),$$

for a sequence $r = r_k \to \infty$.

Part (a) is the $\cos \pi\lambda$-theorem, valid also if $u(r)$ is replaced by

$$A(r) = \inf\{u(z) \colon |z| = r,\, z \in \mathbf{C}\}.$$

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Part (b) is due to A. Beurling ([1]) but, as shown by Hayman [2] and Fryntov (to appear in Proc. Amer. Math. Soc.), the corresponding result for $A(r)$ is false for every $\lambda > 1$. This is also discussed in Chapter 6 of [2]. The following result is in [4]:

Theorem B. (a) *If* $\lambda > 1$, *then*

$$u(r) \geq -B(r) - \frac{\pi^2}{2}\frac{B(r)}{\log B(r)}(1+o(1)), \tag{1}$$

for a sequence $r = r_k \to \infty$.

(b) *If* $\lambda = 1$, *then either* $u(x+iy) = a + bx$ $(b < 0)$ *or*

$$\limsup_{r\to\infty}(u(r) + B(r)) = \infty.$$

The constant $\pi^2/2$ in the inequality (1) is sharp, as shown in [4]:

Theorem C. *Let* $\phi(z) = \sum_{k=0}^{\infty} a_k z^k$, *where* $a_k \geq 0$ *and* $\sum_{k=1}^{\infty} a_k > 0$. *Then if* $u(z) = -\Re \exp \phi(z)$, *we have*

$$u(r) \leq -B(r) - \frac{\pi^2}{2}\frac{B(r)}{\log B(r)}(1+o(1)), \qquad r \to \infty.$$

2. Results

The present note starts from the observation that the functions of Theorem C all have infinite lower order. It turns out that (1) can be improved if $\lambda < \infty$.

Theorem 1. *Let* $u(z)$ *be subharmonic in* $U(\theta)$, $0 < \theta \leq \pi$, *and continuous on the boundary, and suppose further that* $\lambda \geq \pi/\theta$. *If*

$$\limsup_{r\to\infty}(u(r) + B(r)) < \infty, \tag{2}$$

then

$$\int_1^\infty (u(r) + B(r))\frac{\log r}{r^{\lambda+1}}\,dr > -\infty.$$

The surprising thing, perhaps, in Theorem 1 is the presence of the log r factor. But Theorem 1 is sharp, as is shown by

Theorem 2. *Given any* λ *with* $1 < \lambda < \infty$ *and any positive increasing function* $\psi(r) \to \infty$ *as* $r \to \infty$, *there is a function* $u(z)$, *subharmonic in* $\mathbf{C}$ *with order* λ, *satisfying* (2) *and such that*

$$\int_1^\infty (u(r) + B(r))\frac{\psi(r)\log r}{r^{\lambda+1}}\,dx = -\infty. \tag{3}$$

Theorem 2 can give an improved version of Theorem 2 of [3].

Corollary 1. *For any* $\lambda > 1$ *there exists an entire function* $f(z)$ *of perfectly regular growth of order* λ *such that*

$$\int_1^\infty (\log m(r) + \log M(r))\frac{\psi(r)\log r}{r^{1+\lambda}}\,dr = -\infty.$$

Here, of course

$$m(r) = \min\{|f(z)|\colon |z| = r\}, \qquad M(r) = \max\{|f(z)|\colon |z| = r\}$$

and $\psi(r)$ is, as before, an arbitrary function tending to ∞ as $r \to \infty$. It will be clear from the proof of Theorem 5 that (3) can be achieved for $\lambda = 1$, but the corresponding functions are subharmonic only in $U(\pi)$ and not in $\mathbf{C}$. Drasin has shown recently (private communication) that there exists an entire function of order 1 maximal type for which

$$\log m(r) + \log M(r) \to -\infty \qquad (r \to \infty),$$

thus answering a question posed by Hayman in [3]. Because of the above-mentioned difficulty, we are unable to extend Corollary 1 to the case $\lambda = 1$ and maximal type.

We do not give the proof of Corollary 1. Although our construction yields only a subharmonic function, the necessary adjustments to obtain an entire function are precisely those of [3], Section 2.

3. Proof of Theorem 1

It is enough to consider the case $\theta = \pi$ and $\lambda = 1$ (otherwise we consider $v(z) = u(z^{1/\lambda})$ in $U(\pi)$). By subtracting a suitable constant, if necessary, we may also assume, by (2), that

$$u(r) + B(r) < 0, \tag{4}$$

for $r > 0$. For $0 < \varepsilon < \pi/2$ set, for $0 < \phi < \pi - \varepsilon$,

$$u_\varepsilon(re^{i\varphi}) = u(re^{i(\pi-\varepsilon-\varphi)}) + u(re^{i\varphi})$$

so that u_ε is subharmonic in the sector $V(\varepsilon) = \{z \colon 0 < \arg z < \pi - \varepsilon\}$ and continuous on the boundary. Moreover, from (4)

$$u_\varepsilon(r) < 0, \qquad u_\varepsilon(re^{i(\pi-\varepsilon)}) < 0,$$

for all $r > 0$. Since $u_\varepsilon(z) \le 2B(|z|)$, the lower order of u_ε is at most λ (which we have taken equal to 1). So, the Phragmen-Lindelöf Principle yields that $u_\varepsilon(re^{i\varphi}) < 0$, in V_ε. In particular

$$0 > u_\varepsilon(re^{i\frac{\pi-\varepsilon}{2}}) = 2u(re^{i\frac{\pi-\varepsilon}{2}}).$$

Since these inequalities hold for every $0 < \varepsilon < \pi/2$ and every $0 < \varphi < \pi - \varepsilon$, we conclude that $u(z) < 0$, $0 < \arg z < \pi/2$; and that $u_0(re^{i\varphi})) = u(re^{i(\pi-\varphi)}) + u(re^{i\varphi}) < 0$, $0 \le \varphi \le \pi/2$. Similar inequalities hold for $-\pi/2 < \arg z < 0$ for $u(z)$ and for $\pi/2 < \varphi < \pi$ for $u_0(re^{i\varphi})$. Thus

$$u(z) < 0, \quad y = \Im z > 0; \qquad u_0(z) < 0, \quad x = \Re z > 0.$$

From the limiting case $R \to \infty$ of [2], Lemma 6.1, p. 296, the boundary values $u_0(x)$ and $u(iy)$ satisfy

$$-\infty < \int_{-\infty}^{\infty} \frac{u(iy)}{1+y^2}\,dy < \int_0^{\infty} \frac{u(iy)}{1+y^2}\,dy.$$

Analogously, since $u_0(z) < 0$ for $\Re z > 0$, we have

$$-\infty < \int_{-\infty}^{\infty} \frac{u_0(x)}{1+x^2}\,dx.$$

Moreover, from [2], Lemma 6.6, p. 317,

$$2u(iy) = u_0(iy) \le \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{u_0(x)}{x^2+y^2}\,dx = \frac{4y}{\pi}\int_0^{\infty} \frac{u(x)+u(-x)}{x^2+y^2}\,dx.$$

Combining these we obtain, since $u(iy) \le 0$, that

$$\begin{aligned}
-\infty < \int_0^{\infty} \frac{u(iy)}{1+y^2}\,dy &\le \int_1^{\infty} \frac{u(iy)}{2y^2}\,dy \\
&\le \frac{2}{\pi}\int_0^{\infty}\left(\int_{e^{-1}}^{\infty} \frac{1}{y(x^2+y^2)}\,dy\right)(u(x)+u(-x))\,dx \\
&\le \frac{1}{\pi}\int_{e^{-1}}^{\infty} \frac{1+\log x}{x^2+1}(u(x)+u(-x))\,dx,
\end{aligned}$$

which proves Theorem 1.

4. Proof of Theorem 2

We may suppose that $1 < \lambda \le 2$. For if $\lambda > 2$ we may write $\lambda = n\lambda_0$ where $n \in \mathbf{N}$ and $1 < \lambda_0 \le 2$. If $u_0(z)$ is the subharmonic function whose existence is asserted for λ_0 and for which the integral (3) diverges, then $u(z) = u_0(z^n)$ satisfies (3) for a given value λ. The function $u(z)$ will have the form, for $|\phi| \le \pi$,

$$u(re^{i\phi}) = -\Re\int_0^{\lambda-1} z^{\lambda-t}p(t)\,dt, \qquad z = re^{i\phi}, \tag{5}$$

for a suitable function $p(t)$.

For the given function $\psi(t)$ of Theorem 2 we set

$$\begin{aligned}
\psi_1(t) &= \int_0^{\infty} \psi(e^{r/t})re^{-r}\,dr; \\
\psi_2(t) &= \int_0^{\infty} \psi(e^{r/t})e^{-t}\,dr.
\end{aligned}$$

Since $\psi(r) \to \infty$ as $r \to \infty$, we conclude that

$$\begin{aligned}
&\psi_j(t) \to \infty, \qquad t \to 0,\ j = 1, 2; \\
&\psi_1(t) - 4\psi_2(t) \to \infty, \qquad t \to 0.
\end{aligned}$$

We now choose the function $p(t)$ to satisfy the following conditions:

$$p(0) = 0; \qquad p'(t) > 0, \quad 0 < t < \lambda - 1; \qquad \int_0^{\lambda-1} p(t)\,dt = 1;$$

and

$$\int_0^{\lambda-1} p(t)(\psi_1(t) - 4\psi_2(t))\,dt = \infty. \tag{6}$$

Clearly such a function $p(t)$ exists. Now $u(z)$ defined by (5) is subharmonic in $U(\pi)$. Moreover, since $\sin\pi(\lambda - t) < 0$ for $0 < t < \lambda - 1$, we have $(\partial u/\partial\varphi)(-r) < 0$, for all $r > 0$. Using this inequality it is easy to establish that, for sufficiently small ρ,

$$u(-r) \le \frac{1}{2\pi}\int_0^{2\pi} u(-r + \rho e^{i\theta})\,d\theta.$$

Thus $u(z)$ is, in fact, subharmonic in $\mathbf{C}$. We remark that it is precisely here that our argument is unable to deal with the case $\lambda = 1$.

We define $\alpha(r)$, perhaps not uniquely, by the equation

$$B(r) = u(re^{i\alpha(r)}), \qquad r > 0,$$

and note that

$$(\partial u/\partial \varphi)(re^{i\alpha(r)}) = \int_0^{\lambda-1} r^{\lambda-t}(\lambda - t)p(t)\sin(\alpha(r)(\lambda - t))\,dt = 0.$$

Simple arguments prove that $\alpha(r) > \pi/\lambda$ and that $\alpha(r) \to \pi/\lambda$, $r \to \infty$.

We set $\varepsilon(r) = \lambda\alpha(r) - \pi$ so that

$$\begin{aligned} u(r) + B(r) &= -2\int_0^{\lambda-1} r^{\lambda-t}\cos^2\left((\lambda - t)\frac{\pi + \varepsilon(r)}{2\lambda}\right)p(t)\,dt \\ &= -2\int_0^{\lambda-1} r^{\lambda-t}\sin^2\left(\frac{\varepsilon(r)}{2} - \frac{t(\pi + \varepsilon(r))}{2\lambda}\right)p(t)\,dt. \end{aligned}$$

Using the estimates $\sin x > x/2$ for $|x| < \pi/6$ and $|\varepsilon(r)/2 - t(\pi + \varepsilon(r))/2\lambda| < \pi/6$ for $0 < t < \lambda/4$ as $r \to \infty$, we obtain, as $r \to \infty$,

$$\begin{aligned} u(r) + B(r) &\le -\frac{1}{2}\int_0^{\lambda-1} r^{\lambda-t}\left(\frac{\varepsilon(r)}{2} - t\frac{\pi + \varepsilon(r)}{2\lambda}\right)^2 p(t)\,dt + O(r^{\frac{3\lambda}{4}}) \\ &= -\frac{(\pi + \varepsilon(r))^2}{8\lambda^2}\int_0^{\lambda-1} r^{\lambda-t}(t - \delta(r))^2\,dt + O(r^{\frac{3\lambda}{4}}) \\ &\le -\frac{\pi^2}{16\lambda^2}\int_0^{\lambda-1} r^{\lambda-t}(t - \delta(r))^2 p(t)\,dt, \end{aligned}$$

where, for convenience, $\delta(r) = \lambda\varepsilon(r)/(\pi + \varepsilon(r))$.

Inspection shows that the last integral attains its minimum when $\delta(r) = \delta_0(r)$, where

$$\delta_0(r) = \frac{\int_0^{\lambda-1} tr^{-t}p(t)\,dt}{\int_0^{\lambda-1} r^{-t}p(t)\,dt}.$$

Now

$$\int_0^{\lambda-1} tr^{-t}p(t)\,dt = -\left.\frac{tp(t)r^{-t}}{\log r}\right|_0^{\lambda-1} + \frac{1}{\log r}\int_0^{\lambda-1} r^{-t}(p(t) + tp'(t))\,dt,$$

and since $p'(t) > 0$, it follows that $\delta_0(r) < 2/\log r$ as $r \to \infty$. We conclude that

$$u(r) + B(r) \le -\frac{\pi^2}{16\lambda^2}\int_0^{\lambda-1} r^{\lambda-t}\left(t^2 - \frac{4t}{\log r}\right)p(t)\,dt.$$

The proof now follows immediately. From (6) we obtain

$$\begin{aligned}\int_1^\infty &(u(r)+B(r))\frac{\psi(r)\log r}{r^{\lambda+1}}\,dr\\ &\le \frac{\pi^2}{8\lambda^2}\left(-\int_0^\infty\int_0^{\lambda-1}\psi(e^r)re^{-tr}t^2p(t)\,dt\,dr\right.\\ &\qquad\left.+4\int_0^\infty\int_0^{\lambda-1}\psi(e^r)e^{-tr}tp(t)\,dt\,dr\right)\\ &=-\frac{\pi^2}{8\lambda^2}\int_0^{\lambda-1}p(t)(\psi_1(t)-4\psi_2(t))\,dt=-\infty.\end{aligned}$$

Theorem 2 is proved.

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