

MODULI OF NEAR CONVEXITY OF THE BAERNSTEIN SPACE

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ABSTRACT. In this paper we give the exact formulas for the so-called moduli of near convexity of the Baernstein space. This space is frequently used in the geometric theory of Banach spaces and in other branches of nonlinear functional analysis.

1. INTRODUCTION

In the classical geometry of Banach spaces the concepts of *strict convexity* and *uniform convexity* play very important roles and are successfully used in nonlinear functional analysis, operator theory and optimal control theory (cf. [7, 8, 13, 20], for example). Particularly, the concept of the *modulus of convexity* associated with the concept of uniform convexity seems to be a very useful tool that allows us to measure the convexity of Banach spaces and to select a class of spaces having normal structure and being reflexive, for example [10].

In the last years a lot of papers have appeared containing interesting generalizations of the concepts mentioned above. Namely, using the notion of a measure of noncompactness, Huff [12] and, independently, Goebel and Sękowski [11] introduced the concept of *near convexity* in Banach spaces. These papers together with the paper [16] initiated a new subject of the geometry of Banach spaces, which depends on the study of some properties of Banach spaces from the view point of compactness conditions. The central role in this subject is played by the concept of the *modulus of near convexity* defined below. With the help of this modulus we can define several useful classes of Banach spaces being more general than those selected in the classical theory. By this regard the notion of the modulus of near convexity is very applicable in numerous branches of functional analysis (cf. [3, 4, 9, 10, 11, 12, 14, 15, 16, 17, 19] and references therein).

In order to define the basic concepts investigated in this paper let us assume that $(E, \|\cdot\|)$ is an infinite-dimensional Banach space with the zero element θ . Denote by U_E the closed unit ball in E . If X is a set in E , then by

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$\text{conv } X$ and $\text{Conv } X$ we denote the convex hull and the closed convex hull of X , respectively.

Next, let X be a nonempty and bounded subset of E . Recall that the Hausdorff measure of noncompactness $\chi(X)$ of X is defined by

$$\chi(X) = \inf\{\varepsilon > 0: X \text{ admits a finite } \varepsilon\text{-net in } E\}.$$

Similarly, we can define the Kuratowski measure of noncompactness

$$\alpha(x) = \inf\{\varepsilon > 0: X \text{ admits a finite covering by sets of diameters smaller than } \varepsilon\}.$$

For the properties of measures of noncompactness we refer to the monographs [1, 5], for example.

Now, let us recall that the modulus of near convexity of the space E is the function

$$\Delta_E: [0, 1] \rightarrow [0, 1]$$

defined by the formula

$$\Delta_E(\varepsilon) = \inf\{1 - \text{dist}(\theta, X): X \subset U_E, X = \text{Conv } X, \chi(X) \geq \varepsilon\}.$$

Analogously we can define the modulus of near convexity $\tilde{\Delta}_E$ associated with the Kuratowski measure α . Obviously, $\tilde{\Delta}_E: [0, 2] \rightarrow [0, 1]$.

In the case when $\Delta_E(\varepsilon) > 0$ (or $\tilde{\Delta}_E(\varepsilon) > 0$) for $\varepsilon > 0$ we say that E is nearly uniformly convex.

Recall that every nearly uniformly convex Banach space is reflexive and has normal structure [11]. On the other hand there are Banach spaces that are nearly uniformly convex and have bad geometrical structure from the view point of the classical geometry of Banach spaces. For example, the well-known Day space is of such a type [11].

In the next section we show that also the Baernstein space can be considered as the example of a space that is very bad in the classical theory but very nice from the view point of compactness conditions.

2. BAERNSTEIN SPACE AND ITS MODULI OF NEAR CONVEXITY

At the beginning we define the Baernstein space B which will be studied in this section. Denote by \mathcal{A} the set of all sequences (γ_k) of finite subsets of natural numbers such that $\text{card } \gamma_k \leq \min \gamma_k$ and $\max \gamma_k < \min \gamma_{k+1}$ for $k = 1, 2, \dots$. Now, let B be the space of all real sequences $x = (x_n)$ such that

$$\|x\| = \sup \left\{ \left[\sum_{k=1}^{\infty} \left(\sum_{i \in \gamma_k} |x_i| \right)^2 \right]^{1/2} : (\gamma_n) \in \mathcal{A} \right\} < \infty.$$

It was shown in [2] that $(B, \|\cdot\|)$ is a reflexive Banach space which does not possess the Banach-Saks property (cf. also [18]). On the other hand Partington [16] showed that B is nearly uniformly convex without calculating the moduli of near convexity.

Let us emphasize that the Baernstein space B is very useful in this part of geometry of Banach spaces which is connected with the notions described in

Section 1 (cf. [6, 14, 15, 16], for instance). For this reason we will calculate the exact formulas expressing the moduli of near convexity of this space.

In what follows let $e_n = (\delta_{ni})$ ($n = 1, 2, \dots$) be the standard basis of the space B . Write R_n for the projection of B onto the subspace $\overline{\text{lin}}\{e_i: i \geq n+1\}$, i.e., if $x = (x_i) \in B$, then $R_n x = (0, \dots, 0, x_{n+1}, x_{n+2}, \dots)$ and $(I - R_n)x = (x_1, x_2, \dots, x_n, 0, 0, \dots)$. Next, for a fixed $(\gamma_k) \in A$ and $x \in B$ let us denote

$$(\gamma_k)x = \sum_{k=1}^{\infty} \left(\sum_{i \in \gamma_k} |x_i| \right)^2.$$

The following two lemmas will be crucial for our purpose.

Lemma 1 [16]. *For every $x \in B$ and $n \in \mathbb{N}$ the following inequality holds*

$$\|x\|^2 \geq \|(I - R_n)x\|^2 + \|R_n x\|^2.$$

Lemma 2. *Let $x \in B$, $\varepsilon > 0$ and $n \in \mathbb{N}$. Then there exists $q \in \mathbb{N}$, $q > n$, such that*

$$\|x\|^2 \leq \|(I - R_q)x\|^2 + \|R_q x\|^2 + \varepsilon.$$

Proof. Fix $x \in B$, $\varepsilon > 0$ and $n \in \mathbb{N}$. Then we can find $(\gamma_k) \in A$ such that

$$\|x\|^2 \leq (\gamma_k)x + \varepsilon = \sum_{k=1}^{\infty} \left(\sum_{i \in \gamma_k} |x_i| \right)^2 + \varepsilon.$$

On the other hand choose $k_0 \in \mathbb{N}$ such that $\max \gamma_{k_0} > n$. Then, putting $q = \max \gamma_{k_0}$ we have

$$\begin{aligned} \|x\|^2 &\leq \sum_{k=1}^{k_0} \left(\sum_{i \in \gamma_k} |x_i| \right)^2 + \sum_{k=k_0+1}^{\infty} \left(\sum_{i \in \gamma_k} |x_i| \right)^2 + \varepsilon \\ &\leq \|(I - R_q)x\|^2 + \|R_q x\|^2 + \varepsilon, \end{aligned}$$

which ends the proof.

Now, let us observe that the set $\{e_i: i \in \mathbb{N}\}$ forms the standard basis of B . This implies that the Hausdorff measure of noncompactness of a set X , $X \subset B$, can be expressed by the formula [5]

$$\chi(X) = \lim_{n \rightarrow \infty} \{\sup[\|R_n x\|: x \in X]\}.$$

Further, let us fix $\varepsilon > 0$ and choose a closed and convex subset X of U_B such that $\chi(X) \geq \varepsilon$. Without loss of generality we can assume that $\chi(X) > \varepsilon$. Because of $\|R_n x\| \geq \|R_{n+1} x\|$ for any $x \in B$ and $n \in \mathbb{N}$, we have that the sequence of numbers $\sup[\|R_n x\|: x \in X]$ is nonincreasing. Thus there exists a sequence $(x_n) \subset X$ with the property $\|R_n x_n\| \geq \varepsilon$, $n \in \mathbb{N}$. Even more, $\|R_n x_m\| \geq \varepsilon$ for $m \geq n$, $n = 1, 2, \dots$. In view of reflexivity of B we can find a subsequence of (x_n) (which will be denoted also by (x_n)) converging weakly to some $z \in B$.

Next, fix arbitrarily $\delta > 0$. Then, taking into account Lemma 2 we can find $q \in \mathbb{N}$ such that

$$\|z\|^2 \leq \|(I - R_q)z\|^2 + \|R_q z\|^2 + \delta$$

and simultaneously

$$\|R_q z\| \leq \delta.$$

On the other hand for $n \geq q$ we have

$$(1) \quad 1 \geq \|x_n\|^2 \geq \|(I - R_q)x_n\|^2 + \|R_q x_n\|^2 \geq \|(I - R_q)x_n\|^2 + \varepsilon^2.$$

Since $(I - R_q)x_n$ converges weakly to $(I - R_q)z$, we get

$$\liminf_{n \rightarrow \infty} \|(I - R_q)x_n\| \geq \|(I - R_q)z\|.$$

Consequently, by virtue of (1) we obtain

$$1 \geq \|(I - R_q)z\|^2 + \varepsilon^2 \geq \|z\|^2 - \|R_q z\|^2 - \delta + \varepsilon^2 \geq \|z\|^2 - \delta^2 - \delta + \varepsilon^2.$$

Because δ was chosen arbitrary, this implies

$$1 - \varepsilon^2 \geq \|z\|^2.$$

Combining the above inequality and the fact that $z \in X$ we have

$$\sqrt{1 - \varepsilon^2} \geq \text{dist}(\theta, X).$$

This yields

$$\sqrt{1 - \varepsilon^2} \geq \sup\{\text{dist}(\theta, X) : X = \text{Conv } X, X \subset U_B, \chi(X) \geq \varepsilon\}$$

and finally

$$(2) \quad 1 - \sqrt{1 - \varepsilon^2} \leq \Delta_B(\varepsilon).$$

For further purposes let us consider the vectors in B of the form

$$u_n = \sqrt{1 - \varepsilon^2} e_1 + \frac{\varepsilon}{n} \sum_{i=1}^n \frac{1}{2^i} (e_{2^i} + e_{2^{i+1}} + \cdots + e_{2^{i+1}-1}),$$

$n = 1, 2, \dots$. The following lemma will be needed in what follows.

Lemma 3. For every $n \in \mathbb{N}$ the equality

$$\|u_n\| = ((\gamma'_k)u_n)^{1/2}$$

holds, where $\gamma'_k = \{2^{k-1}, 2^{k-1} + 1, \dots, 2^k - 1\}$ for $k = 1, 2, \dots, n+1$ and γ'_k is arbitrary for $k \geq n+2$.

Proof. Put $A_1 = \{(\gamma_k) \in A : \|u_n\| = ((\gamma_k)u_n)^{1/2}\}$. Then $A_1 \neq \emptyset$ because u_n is representable with the help of a finite number of vectors of the basis.

Suppose there exists $n \in \mathbb{N}$ such that

$$\|u_n\| > ((\gamma'_k)u_n)^{1/2}.$$

Then $(\gamma'_k) \notin A_1$ and we can define the mapping

$$T : A_1 \rightarrow \mathbb{N}$$

by putting

$$T((\gamma_k)) = \min \left(\bigcup_{i=1}^{n+1} (\gamma'_i \setminus \gamma_i) \right).$$

Next, let us choose $(\beta_k) \in A_1$ such that

$$(3) \quad T((\beta_k)) = \max\{T((\gamma_k)) : (\gamma_k) \in A_1\}.$$

Denote the above number by p . Then there is a number s , $s \in \{1, 2, \dots, n+1\}$, such that $2^{s-1} \leq p \leq 2^s - 1$. Now, let

$$r = \min\{i \in \mathbb{N}: \beta_i \cap \{2^{n+1}, 2^{n+1} + 1, \dots\} \neq \emptyset\}.$$

Define (β'_k) as

$$\beta'_k = \{2^{k-1}, 2^{k-1} + 1, \dots, 2^k - 1\} \quad \text{for } k = 1, 2, \dots, s$$

and

$$\beta'_k = \beta_{k+r-s-1} \cap \{2^{n+1}, 2^{n+1} + 1, \dots\} \quad \text{for } k \geq s+1.$$

Then $\beta'_k = \beta_k$ for $k = 1, 2, \dots, s-1$ and $\beta'_k = \beta_{k+r-s-1}$ for $k \geq s+2$. Moreover, we have

$$\left(\sum_{i \in \beta'_s} |u_n^i| \right)^2 + \left(\sum_{i \in \beta'_{s+1}} |u_n^i| \right)^2 \geq \sum_{k=s}^r \left(\sum_{i \in \beta_k} |u_n^i| \right)^2,$$

where we have denoted

$$u_n = (u_n^1, u_n^2, \dots).$$

This means that $(\beta'_k)u_n \geq (\beta_n)u_n$. Now, we have two possibilities:

1. $(\beta'_k)u_n > (\beta_k)u_n = \|u_n\|^2$. But this contradicts the definition of the norm in B .

2. $(\beta'_k)u_n = (\beta_k)u_n$. In this case $(\beta'_k) \in A_1$ and consequently

$$T((\beta'_k)) \geq 2^s > T((\beta_k)),$$

which contradicts (3). Thus the proof is complete.

Now, let us consider the set $Y = \{y_n: n \in \mathbb{N}\}$, where

$$y_n = \sqrt{1 - \varepsilon^2} e_1 + \varepsilon e_{n+1}$$

for $n = 1, 2, \dots$. Put $X = \text{Conv } Y$. Then $\chi(X) = \varepsilon$.

In view of the equality $\text{dist}(\theta, X) = \text{dist}(\theta, \text{conv } Y)$ and taking into account that every element $w \in \text{conv } Y$ has the form

$$w = \sqrt{1 - \varepsilon^2} e_1 + \varepsilon \sum_{i=1}^k \alpha_i e_i$$

where $\alpha_i \geq 0$ and $\sum_{i=1}^k \alpha_i = 1$, we infer that $\|w\| \geq \sqrt{1 - \varepsilon^2}$. This implies

$$\text{dist}(\theta, X) \geq \sqrt{1 - \varepsilon^2}.$$

On the other hand observe that the set X contains the vectors u_n ($n = 1, 2, \dots$) defined before. According to Lemma 3 we have that

$$\|u_n\| = ((\gamma'_k)u_n)^{1/2}$$

for $n = 1, 2, \dots$. This implies

$$\|u_n\| = ((\gamma'_k)u_n)^{1/2} = (1 - \varepsilon^2 + \varepsilon^2/n)^{1/2} \rightarrow \sqrt{1 - \varepsilon^2}$$

when $n \rightarrow \infty$. Consequently we get

$$\text{dist}(\theta, X) = \sqrt{1 - \varepsilon^2}$$

and finally

$$\Delta_B(\varepsilon) \leq 1 - \sqrt{1 - \varepsilon^2}.$$

This inequality in conjunction with (2) yields

$$(4) \quad \Delta_B(\varepsilon) = 1 - \sqrt{1 - \varepsilon^2}, \quad \varepsilon \in [0, 1].$$

Now, let us direct our attention to the modulus of noncompact convexity $\tilde{\Delta}_B$ associated with the Kuratowski measure of noncompactness α . In view of a result established in [3] we have

$$(5) \quad \tilde{\Delta}_B(\varepsilon) \geq \Delta_B(\varepsilon/2) = 1 - \sqrt{1 - (\varepsilon/2)^2}$$

for $\varepsilon \in [0, 2]$. On the other hand consider the set $G = \text{Conv}\{g_n: n = 1, 2, \dots\}$, where

$$g_n = \sqrt{1 - \frac{\varepsilon^2}{4}}e_1 + \frac{\varepsilon}{2}e_{n+1}$$

for $n = 1, 2, \dots$. Then, by the same reasoning as above we can infer

$$\text{dist}(\theta, G) = \sqrt{1 - \varepsilon^2/4}.$$

Apart from this it is easy to verify that $\|g_n - g_m\| = \varepsilon$ for $n \neq m$. This implies (cf. [1]) that $\alpha(G) \geq \varepsilon$.

Hence, keeping in mind (5) we obtain

$$(6) \quad \tilde{\Delta}_B(\varepsilon) = 1 - \sqrt{1 - \varepsilon^2/4}, \quad \varepsilon \in [0, 2].$$

Finally, we summarize the results obtained above in the following theorem.

Theorem. *The moduli of near convexity of the Baernstein space B can be expressed by the formulas (4) and (6).*

Observe that in virtue of the above theorem we infer that the Baernstein space B is nearly uniformly convex. This allows us to deduce [11] that it is reflexive and has normal structure.

Moreover, let us notice that from the view point of the “compactness geometry” of Banach spaces the structure of the Baernstein space is similar to the structure of a Hilbert space. Indeed, the modulus of near convexity Δ_B expressed by (4) is exactly the same as the modulus of convexity of a Hilbert space in the classical sense [10].

On the other hand let us mention that the Baernstein space B has very bad geometrical structure from the classical view point. In fact, it is easily seen that the segment joining the points e_n and e_{n+1} lies in the unit sphere of the space B . Thus B is not a strictly convex space; consequently it is not uniformly convex in the classical sense (cf. [13]).

REFERENCES

1. R. R. Akmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodkina, and B. N. Sadovskii, *Measures of noncompactness and condensing operators*, Birkhäuser Verlag, Basel, Boston and Berlin, 1992.
2. A. Baernstein II, *On reflexivity and summability*, *Studia Math.* **42** (1972), 91–94.

3. J. Banaś, *On modulus of noncompact convexity and its properties*, Canad. Math. Bull. **30** (1987), 186–192.
4. ———, *Compactness conditions in the geometric theory of Banach spaces*, Nonlinear Anal. T.M.A. **16** (1991), 669–682.
5. J. Banaś and K. Goebel, *Measures of noncompactness in Banach spaces*, Lecture Notes in Pure and Appl. Math., vol. 60, Marcel Dekker, New York and Basel, 1980.
6. Bor-Luh Lin and Pei-Kee Lin, *A fully convex Banach space which does not have the Banach-Saks property*, J. Math. Anal. Appl. **117** (1986), 273–283.
7. M. M. Day, *Normed linear spaces*, Ergeb. Math. Grenzgeb. (3), vol. 21, Springer-Verlag, Berlin and New York, 1973.
8. K. Deimling, *Nonlinear functional analysis*, Springer-Verlag, Berlin, 1985.
9. T. Dominguez Benavides and J. M. Ayerbe, *Set-contractions and ball-contractions in L^p -spaces*, J. Math. Anal. Appl. **159** (1991), 500–507.
10. K. Goebel and W. A. Kirk, *Topics in metric fixed point theory*, Cambridge Univ. Press, Cambridge, 1990.
11. K. Goebel and T. Sękowski, *The modulus of noncompact convexity*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **38** (1984), 41–48.
12. R. Huff, *Banach spaces which are nearly uniformly convex*, Rocky Mountain J. Math. **10** (1980), 743–749.
13. G. Köthe, *Topological vector spaces I*, Springer-Verlag, Berlin, 1969.
14. D. Kutzarova, *Every (β) -space has the Banach-Saks property*, C. R. Acad. Bulgar. Sci. **42** (1989), 9–12.
15. D. Kutzarova, Bor-Luh Lin, and Wenyao Zhang, *Some geometrical properties of Banach spaces related to nearly uniform convexity*, Contemp. Math., vol. 144, Amer. Math. Soc., Providence, RI, 1993, pp. 165–171.
16. J. R. Partington, *On nearly uniformly convex Banach spaces*, Math. Proc. Cambridge Philos. Soc. **93** (1983), 127–129.
17. S. Rolewicz, *On drop property*, Studia Math. **85** (1987), 27–35.
18. C. J. Seifert, *The dual of Baernstein space and the Banach-Saks property*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **26** (1978), 237–239.
19. T. Sękowski and A. Stachura, *Noncompact smoothness and noncompact convexity*, Atti Sem. Mat. Fis. Univ. Modena **36** (1988), 329–338.
20. M. M. Vainberg, *Variational methods in the study of nonlinear operators*, Holden-Day, San Francisco, 1964.

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