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# A COUNTEREXAMPLE TO THE "FINE" PROBLEM IN PLURIPOTENTIAL THEORY

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ABSTRACT. In this paper we prove that pointwise values of the non-regularized pluriharmonic measure are not capacities. This answers the question raised by E. Bedford and U. Cegrell.

The purpose of this note is to give a counterexample to a problem posed in [1, p. 84] and [2, p. 52].

Let $D$ be a domain in $\mathbb{C}^n$, $A \subset D$, and $z \in D$. Then the function

$$\omega(z, Aa, D) = \sup\{u(z) \colon u < 0 \text{ is plurisubharmonic on } D \text{ and } u \leq -1 \text{ on } A\}$$

is called the *pluriharmonic measure of* $A$ *in* $D$.

The "fine" problem is to decide if the set function taking $A$ to $-\omega(z, A, D)$ is a Choquet capacity on subsets of $D$. The axioms to be checked are:

(1) If $K_j$ is a decreasing sequence of compact sets in $D$, then

$$\lim_{j\to\infty} \omega(z, K_j, D) = \omega\left(z, \bigcap_{j=1}^{\infty} K_j, D\right).$$

(2) If $A_j$ is an increasing sequence of subsets of $D$, then

$$\lim_{j\to\infty} \omega(z, A_j, D) = \omega\left(z, \bigcup_{j=1}^{\infty} A_j, D\right).$$

It is not hard to see that the first axiom is true, since we are looking at the unregularized pluriharmonic measure. Also it was proved in [3] that (2) is true if the sets $A_j$, $j \in \mathbb{N}$, and $A = \bigcup_{j=1}^{\infty} A_j$ are compact.

Using some ideas from [4], adapted to this concrete situation, we are going to construct an $F_\sigma$-set $A$ for which (2) is false, thus giving a counterexample to the "fine" problem.

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Let

$$D = \{z = (z_1, z_2) \in \mathbb{C}^2 : \|z\|^2 = |z_1|^2 + |z_2|^2 < 4\},$$

$$\gamma_0 = \left\{(e^{i\theta}, 0) \in \mathbb{C}^2 : \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\right\},$$

$$\gamma_j = \left\{(e^{i\theta}, 2^{-j}e^{i\theta}) \in \mathbb{C}^2 : -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\},$$

$$A_k = \bigcup_{j=0}^{k} \gamma_j, \qquad A = \bigcup_{j=0}^{\infty} \gamma_j.$$

**Theorem.** $\lim_{k\to\infty} \omega(0, A_k, D) > \omega(0, A, D) = -1$.

*Proof*. Let $u < 0$ be a plurisubharmonic function on $D$ such that $u \le -1$ on $A$. We fix $\varepsilon > 0$ and let $V = \{z \in D : u(z) < -1+\varepsilon\}$. If $j$ is sufficiently large the circles $S_j = \{(\zeta, 2^{-j}\zeta) : |\zeta| = 1\}$ are in $V$. Thus $u(0) \le -1+\varepsilon$ by the maximum principle. Therefore $\omega(0, A, D) = -1$.

Denote by $\Gamma_0 = \{(z_1, 0) \in D\}$ and $\Gamma_j = \{(z_1, 2^{-j}z_1) \in D\}$. Then $\Gamma_0$, $\Gamma_j$, $j \in \mathbb{N}$, are complete pluripolar sets in $D$, and $\gamma_0 \subset \Gamma_0$, $\gamma_j \subset \Gamma_j$, $j \in \mathbb{N}$. We claim that there is a constant $a$, $0 > a > -1$, such that for $\|z\| = 1/16$ we have $\omega(z, A, D) \ge a$.

Note first that if $z \in D$ and $z \notin \bigcup_{j=0}^{\infty} \Gamma_j$, then $\omega(z, A, D) = 0$. Consider the function $v(z) = \omega(z, K, R)$ where $R$ is the disk in $\mathbb{C}$ of radius 3 centered at 0 and $K = \{e^{i\theta} : -\pi/2 \le \theta \le \pi/2\}$. Since $K$ is polynomially convex,

$$a = \inf_{|z|\le 1/4} \omega(z, K, R) > -1.$$

Therefore, $v_j(z) = v(z_1(1+2^{-j}) - z_2) \ge a$ when $z = (z_1, z_2) \in \Gamma_j$, $j \ge 1$, and $\|z\| \le 1/16$. It is easy to check that $v_j = -1$ on $\gamma_j$. When $j = 0$ we consider $v_0(z_1, z_2) = v(-z_1)$. Clearly $v_0$ satisfies the properties above.

Suppose that $z_0 \in \Gamma_m$ and $\|z_0\| = 1/16$. Then $z_0 \notin \Gamma^m := \bigcup_{j\ne m} \Gamma_j$, so since each $\Gamma_j$ is a complete pluripolar set, for every $\varepsilon > 0$ we can find a negative plurisubharmonic function $g$ on $D$ such that $g = -\infty$ on $\Gamma^m$ but $g(z_0) > -\varepsilon$. Since $g(z) + v_m(z) \le -1$ on $A$, we see that $\omega(z_0, A, D) \ge a - \varepsilon$ and, since $\varepsilon > 0$ is arbitrary, $\omega(z_0, A, D) \ge a$.

We now prove that

$$\omega(0, A_k, D) \ge a$$

for any $k$. Since $A_k$ is compact in $D$,

$$\omega(z, A_k, D) = \sup\{u(z) : u < 0 \text{ is plurisubharmonic and continuous in } D \text{ and } u \le -1 \text{ on } A_k\}.$$

Therefore, for every $\varepsilon > 0$ there is a continuous plurisubharmonic function $u$ on $D$ such that $u \le 0$ on $D$, $u \le -1$ on $A_k$, and $u(z) \ge a - \varepsilon$ when $\|z\| = 1/16$. Define $h(z) = u(z)$ when $\|z\| \ge 1/16$ and $h(z) = \max(u(z), a-\varepsilon)$ when $\|z\| < 1/16$. Then $h(z)$ is negative and plurisubharmonic on $D$ and $h \le -1$ on $A_k$. Thus $\omega(z, A_k, D) \ge h(z)$, so in particular $\omega(0, A_k, D) \ge h(0) \ge a - \varepsilon$, which completes the proof of the theorem. $\square$

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