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# THE BEREZIN SYMBOL AND MULTIPLIERS OF FUNCTIONAL HILBERT SPACES

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ABSTRACT. This paper focuses on a multiplicative property of the Berezin symbol $\widetilde{A}$, of a given linear map $A: \mathscr{H} \mapsto \mathscr{H}$, where $\mathscr{H}$ is a functional Hilbert space of analytic functions. We show $\widetilde{AB} = \widetilde{A}\widetilde{B}$ for all $B$ in $\mathscr{B}(\mathscr{H})$ if and only if $A$ is a multiplication operator $M_\varphi$, where $\varphi$ is a multiplier. We also present a version of this result for vector-valued functional Hilbert spaces.

## 1. INTRODUCTION

Let $n$ be a fixed positive integer and let $\Omega$ be a region in $\mathbf{C}^n$. A functional Hilbert space $\mathscr{H}$ is a Hilbert space of analytic functions on $\Omega$ such that the point evaluations are bounded, linear functionals. By the Riesz-representation theorem there exists, for each $z$ in $\Omega$, a unique element $K_z$ of $\mathscr{H}$ such that $f(z) = \langle f, K_z\rangle$ for all $f$ in $\mathscr{H}$. The function $K$ on $\Omega \times \Omega$, defined by $K(z, w) = K_w(z)$, is called the reproducing kernel function of $\mathscr{H}$. Let $k_z = \frac{K_z}{\|K_z\|}$ be the normalized reproducing kernel function. For a given linear map $A: \mathscr{H} \mapsto \mathscr{H}$, the Berezin symbol $\widetilde{A}$ (see [1]) of a map $A$ of $\mathscr{H}$ into itself is defined by

$$\widetilde{A}(z) = \langle Ak_z, k_z\rangle.$$

It is known that the map $A \mapsto \widetilde{A}$ is injective (see [3]). A function $\varphi$ defined on $\Omega$ is a multiplier of $\mathscr{H}$ if $\varphi \cdot f$ is in $\mathscr{H}$, for all $f$ in $\mathscr{H}$. Let $\mathscr{B}(\mathscr{H})$ denote the set of all bounded, linear operators from $\mathscr{H}$ into $\mathscr{H}$. The multiplication operator $M_\varphi: \mathscr{H} \mapsto \mathscr{H}$ defined by $M_\varphi f = \varphi \cdot f$ is in $\mathscr{B}(\mathscr{H})$, when $\varphi$ is a multiplier of $\mathscr{H}$.

## 2. THE MULTIPLICATIVE PROPERTY OF THE BEREZIN SYMBOL ON A FUNCTIONAL HILBERT SPACE

**Theorem 1.** *Let $A$ be a bounded operator on $\mathscr{H}$. Then*

$$\widetilde{AB}(z) = \widetilde{A}(z)\widetilde{B}(z)$$

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*for all $B$ in $\mathscr{B}(\mathscr{H})$ if and only if $A$ is a multiplication operator, $M_\varphi$, where $\varphi$ is a multiplier. Moreover, $\varphi = \widetilde{A}$.*

Before proceeding with the proof, we need the following:

**Lemma 1.** *When $\varphi$ is a multiplier of $\mathscr{H}$, $\widetilde{M_\varphi}(z) = \varphi(z)$.*

*Proof.* $\widetilde{M_\varphi}(z) = \langle M_\varphi k_z, k_z\rangle = \langle \varphi k_z, k_z\rangle = \varphi(z)$.

**Lemma 2.** *The Berezin symbol of $f \otimes g$, for $f$, $g$ in $\mathscr{H}$, is*

$$(\widetilde{f\otimes g})(z) = \frac{\overline{g(z)}}{\|K_z\|^2} f(z), \qquad z \in \Omega.$$

*Proof.* For $f$ and $g$ in $\mathscr{H}$ and $z$ in $\Omega$,

$$\begin{aligned}(\widetilde{f\otimes g})(z) &= \left\langle (f\otimes g)\frac{K_z}{\|K_z\|} \frac{K_z}{\|K_z\|}\right\rangle \\ &= \frac{1}{\|K_z\|^2}\langle K_z, g\rangle\langle f, K_z\rangle.\end{aligned}$$

By the reproducing property of the kernel function, we have

$$(\widetilde{f\otimes g})(z) = \frac{\overline{g(z)}}{\|K_z\|^2} f(z), \qquad f, q \in \mathscr{H}.$$

*Proof of Theorem* 1. Suppose $\widetilde{AB}(z) = \widetilde{A}(z)\widetilde{B}(z)$ for all $B$ in $\mathscr{B}(\mathscr{H})$. Let $B = f \otimes g$ for $f$ and $g$ in $\mathscr{H}$. Then, by Lemma 2,

$$\widetilde{AB}(z) = (\widetilde{Af\otimes g})(z) = \frac{\overline{g(z)}}{\|K_z\|^2}(Af)(z).$$

By the hypothesis, we have

$$\frac{\overline{g(z)}}{\|K_z\|^2}(Af)(z) = \frac{\overline{g(z)}}{\|K_z\|^2}\widetilde{A}(z) f(z),$$

which reduces to

$$(Af)(z) = \widetilde{A}(z) f(z)$$

for all $f$ in $\mathscr{H}$. Hence $A = M_{\widetilde{A}}$.

Conversely, if $A$ is a multiplication operator, $M_\varphi$, where $\varphi$ is a multiplier,

$$\widetilde{M_\varphi B} = \langle M_\varphi B k_z, k_z\rangle = \varphi(z)\frac{(Bk_z)(z)}{\|K_z\|}$$

for all $B$ in $\mathscr{B}(\mathscr{H})$. By Lemma 1, we have

$$\widetilde{M_\varphi B}(z) = \widetilde{M_\varphi}(z)\widetilde{B}(z)$$

for all $B$ in $\mathscr{B}(\mathscr{H})$.

**Corollary 1.** *Let $B$ be in $\mathscr{B}(\mathscr{H})$. Then*

$$\widetilde{AB}(z) = \widetilde{A}(z)\widetilde{B}(z)$$

*for all $A$ in $\mathscr{B}(\mathscr{H})$ if and only if $B = M_\psi^*$, where $\psi$ is a multiplier.*

*Proof.* The assertion follows from Theorem 1 and the fact that $\widetilde{T^*}(z) = \overline{\widetilde{T}}(z)$, for all $T$ in $\mathscr{B}(\mathscr{H})$.

The Hardy space $H^2$ consists of the complex-valued analytic functions on the unit disk $\mathbf{D}$ such that the Taylor coefficients are square summable. A calculation shows that $K_z = \frac{1}{1-\bar{z}w}$ has the reproducing property (see [4]). Let $P$ denote the orthogonal projection of $L^2(\partial\mathbf{D})$ onto $H^2$, and let $\varphi$ be a bounded measurable function. Then the Toeplitz operator, $T_\varphi$, induced by $\varphi$ is defined by $T_\varphi f = P(\varphi f)$, for all $f$ in $H^2$.

**Corollary 2.** *Let $A$ be a bounded operator on $H^2$. Then*

$$\widetilde{AB}(z) = \widetilde{A}(z)\widetilde{B}(z)$$

*for all $B$ in $\mathscr{B}(H^2)$ if and only if $A$ is a Toeplitz operator, $T_\varphi$, induced by $\varphi$ in $H^\infty$. Moreover $\varphi = \widetilde{A}$.*

*Proof.* The multiplication operators on $H^2$ are the analytic Toeplitz operators.

We should mention that Corollary 2 is also true if one replaces $H^2$ by the Bergman space or any of the weighted Bergman spaces. (For analytic Toeplitz operators on weighted Bergman spaces see [6].)

## 3. The multiplicative property of the Berezin symbol on the analytic reproducing kernel space, $\mathscr{H} = \mathscr{H}_0 \otimes \mathscr{C}$

Let $\mathscr{H}_0$ be a functional Hilbert space of (scalar-valued) analytic functions on $\Omega$ with the reproducing kernel function $K_z$, for each fixed $z$ in $\Omega$. Let $\mathscr{C}$ be a separable Hilbert space, and let $\mathscr{H}$ be the functional Hilbert space of $\mathscr{C}$-valued functions, $\mathscr{H} = \mathscr{H}_0 \otimes \mathscr{C}$. The reproducing kernel function of $\mathscr{H}$, $\mathfrak{J}_z \colon \mathscr{C} \mapsto \mathscr{H}$, is defined by $\mathfrak{J}_z(u) = K_z \otimes u$, where $u$ is in $\mathscr{C}$.

The evaluation functional $E_z \colon \mathscr{H} \mapsto \mathscr{C}$, defined by $E_z f = f(z)$, for $z$ in $\Omega$, is bounded (see [2], Lemma 3.2). For $f \in \mathscr{H}$, $u$ in $\mathscr{C}$, we have

$$\langle f, E_z^* u\rangle_{\mathscr{H}} = \langle f(z), u\rangle_{\mathscr{C}}.$$

We also have the reproducing property of the kernel function, that is

$$\langle f, \mathfrak{J}_z(u)\rangle_{\mathscr{H}} = \langle f(z), u\rangle_{\mathscr{C}}.$$

Therefore, $E_z^* u = \mathfrak{J}_z(u)$, for all $u$ in $\mathscr{C}$. By the reproducing property of the kernel function, we have $\|\mathfrak{J}_z(u)\|^2 = K_z(z)\|u\|^2$, where $u$ is in $\mathscr{C}$, and hence $\|\mathfrak{J}_z\| = \sqrt{K_z(z)} = \|E_z\|$.

Let $\mathscr{K}_z = \frac{\mathfrak{J}_z}{\|\mathfrak{J}_z\|}$ be the normalized reproducing kernel function, and let $A$ be a bounded linear operator on $\mathscr{H}$. Then the Berezin symbol $\widetilde{A}$ of $A$ is defined by

$$\widetilde{A}(z) = \mathscr{K}_z^* A \mathscr{K}_z.$$

**Lemma 3.** *An operator $A$ is a multiplication operator if and only if, for each fixed $z$ in $\Omega$, $A^* E_z^* = E_z^* \Phi(z)^*$ for some operator $\Phi(z)$ in $\mathscr{B}(\mathscr{C})$. Moreover, in this case, $A$ is the operator of multiplication by the function $z \mapsto \Phi(z)$.*

*Proof.* Let $z$ be fixed in $\Omega$. Suppose $A$ is a multiplication operator, $M_\Phi$, induced by $\Phi \colon \Omega \to \mathscr{B}(\mathscr{C})$. We observe that

$$E_z M_\Phi f = M_\Phi f(z) = \Phi(z) f(z) = \Phi(z) E_z f \quad \text{for all } f \text{ in } \mathscr{H}.$$

Then we have $E_z M_\Phi = \Phi(z) E_z$, for some operator $\Phi(z)$ in $\mathscr{B}(\mathscr{C})$.

Conversely, let $A$ be a bounded operator on $\mathscr{H}$ such that $A^*E_z^* = E_z^*\Phi(z)^*$ for some operator $\Phi(z)$ in $\mathscr{B}(\mathscr{E})$. For $u$ in $\mathscr{E}$, we have

$$\langle f, A^*E_z^*u\rangle_{\mathscr{H}} = \langle Af, E_z^*u\rangle_{\mathscr{H}} = \langle (Af)(z), u\rangle_{\mathscr{E}} \quad \text{for all } f \text{ in } \mathscr{H}.$$

On the other hand, for $u$ in $\mathscr{E}$, we have $\langle f, E_z^*\Phi(z)^*u\rangle = \langle \Phi(z)f(z), u\rangle$, for all $f$ in $\mathscr{H}$. Then $\langle (Af)(z), u\rangle = \langle \Phi(z)f(z), u\rangle$, for all $f$ in $\mathscr{H}$ and $u$ in $\mathscr{E}$. Therefore, $(Af)(z) = \Phi(z)f(z)$, for all $f$ in $\mathscr{H}$.

**Theorem 2.** *Let $A$ be a bounded operator on $\mathscr{H}$. Then*

$$\widetilde{AB}(z) = \widetilde{A}(z)\widetilde{B}(z)$$

*for all $B$ in $\mathscr{B}(\mathscr{H})$ if and only if $A = M_\Phi$, where $\Phi\colon \Omega \mapsto \mathscr{B}(\mathscr{E})$.*

*Proof.* We observe that $E_zM_\Phi f = \Phi(z)f(z)$, for all $f$ in $\mathscr{H}$. Then $E_zM_\Phi E_z^* = \Phi(z)E_zE_z^*$ and $E_zM_\Phi BE_z^* = \Phi(z)E_zBE_z^*$, for all $B$ in $\mathscr{B}(\mathscr{H})$. Since $E_zE_z^* = K_z(z)I_{\mathscr{E}}$, we have $\widetilde{M_\Phi} = \Phi(z)$ and

$$\widetilde{M_\Phi B}(z) = \Phi(z)\frac{E_zBE_z^*}{\|\mathfrak{I}_z\|^2} = \widetilde{M_\Phi}(z)\widetilde{B}(z) \quad \text{for all } B \text{ in } \mathscr{B}(\mathscr{H}).$$

Conversely, suppose that $A$ is a bounded operator such that $\widetilde{AB}(z) = \widetilde{A}(z)\widetilde{B}(z)$ for all $B$ in $\mathscr{B}(\mathscr{H})$. Then from the definitions, we get

$$E_zABE_z^* = \frac{1}{\|E_z\|^2}E_zAE_z^*E_zBE_z^* \quad \text{for all } B \text{ in } \mathscr{B}(\mathscr{H}).$$

For $u$ and $v$ in $\mathscr{E}$, we have

$$\langle E_zABE_z^*u, v\rangle = \left\langle \frac{E_zAE_z^*}{\|E_z\|^2}E_zBE_z^*u, v\right\rangle = \langle \widetilde{A}(z)E_zBE_z^*u, v\rangle.$$

Then we have

$$\langle BE_z^*u, A^*E_z^*v\rangle = \langle BE_z^*u, E_z^*\widetilde{A}(z)^*v\rangle.$$

For each fixed nonzero $u$, $BE_z^*u$ runs through all vectors in $\mathscr{H}$ as $B$ runs through all elements of $\mathscr{B}(\mathscr{H})$. Thus we see that $A^*E_z^* = E_z^*\widetilde{A}(z)^*$, for all $z$ in $\Omega$. Therefore $A$ is a multiplication operator, $M_{\widetilde{A}}$, by Lemma 3.

Let us note that if we take $\mathscr{E}$ to be $\mathbf{C}$ and define $\mathscr{K}_z = k_z \otimes 1$, the sufficiency proof of Theorem 2 will also work for Theorem 1, the scalar-valued case.

Let $\mathbf{N} = \{0, 1, 2, \ldots\}$ denote the set of nonnegative integers. The set $\mathbf{N}^n$ is partially ordered by setting $\mathbf{I} = (i_1, i_2, \ldots, i_n) \geq (j_1, j_2, \ldots, j_n) = \mathbf{J}$ if and only if $i_k \geq j_k$ for $k = 1, 2, \ldots, n$. If $z = (z_1, z_2, \ldots, z_n)$ is in $\Omega$, then we set $z^I = z_1^{i_1} \cdot z_2^{i_2} \cdot \cdots \cdot z_n^{i_n}$. We denote by $H^2(n) \otimes \mathscr{E}$, where $H^2(n) = H^2 \otimes H^2 \otimes \cdots \otimes H^2$ ($n$ copies), the set of all vector-valued analytic functions $f\colon \mathbf{D}^n \mapsto \mathscr{E}$ with power series expansion $f(z) = \sum_{I \in \mathbf{N}^n} z^I v_I$, with $v_I$ in $\mathscr{E}$ and $z$ in $\mathbf{D}^n$, such that $\sum_{I\in\mathbf{N}^n} \|v_I\|_{\mathscr{E}}^2 < \infty$.

The space $H^2(n)\otimes\mathscr{E}$ is a Hilbert space with the reproducing kernel function, $\mathfrak{I}_z\colon \mathscr{E} \mapsto H^2(n) \otimes \mathscr{E}$, for $z$ in $\mathbf{D}^n$, defined by $\mathfrak{I}_z(u) = K_z \otimes u$, where $u$ is in $\mathscr{E}$ and $K_z(w) = \sum_{I\in\mathbf{N}^n} \overline{z}^I w^I$ is the reproducing kernel function for $H^2(n)$ (see [5]). Let $H^\infty(n)(\mathscr{B}(\mathscr{E}))$ denote the Banach space of all bounded analytic functions $\Phi\colon \mathbf{D}^n \mapsto \mathscr{B}(\mathscr{E})$ with the norm $\|\Phi\|_\infty = \sup\{\|\Phi(z)\|, \text{ for } z \in \mathbf{D}^n\}$.

For every $\Phi$ in $H^\infty(n)(\mathscr{B}(\mathscr{C}))$, we can define the analytic Toeplitz operator $T_\Phi$ in $\mathscr{B}(H^2(n)\otimes\mathscr{C})$ as follows:

$$(T_\Phi f)(z) = \Phi(z)f(z), \qquad z \text{ in } \mathbf{D}^n,\ f \text{ in } H^2(n)\otimes\mathscr{C}.$$

For the boundedness of the map $T_\Phi$ see [2].

**Corollary 3.** *Let $A$ be a bounded operator on $H^2(n)\otimes\mathscr{C}$. Then*

$$\widetilde{AB}(z) = \widetilde{A}(z)\widetilde{B}(z)$$

*for all $B$ in $\mathscr{B}(H^2(n)\otimes\mathscr{C})$ if and only if $A = T_\Phi$, where $\Phi$ is in $H^\infty(n)(\mathscr{B}(\mathscr{C}))$.*

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