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ISOLATED SINGULARITIES OF MONGE-AMPÈRE EQUATIONS

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ABSTRACT. In this paper, we give conditions which ensure that isolated singularities of solutions of the elliptic Monge-Ampère equation $\det D^2u = 1$ are removable.

INTRODUCTION

We use the notation D^2u for the matrix of second-order derivatives of the function $u = u(x)$. Furthermore, for a point x in $\mathbf{R}^n$, $|x|$ is the norm, $e_1, \dots, e_n$ are the unit vectors in the direction of the coordinate axis, and $B_r = \{x \colon |x| < r\}$ is the ball of radius r centered at the origin 0.

For the statement of the main theorem of this paper, let u be defined in the punctured ball $B_2\backslash\{0\}$.

Theorem 1. *Suppose that u is a smooth convex solution of the elliptic Monge-Ampère equation*

$$\det D^2u = 1$$

in $B_2\backslash\{0\}$. Then u has a locally Lipschitz continuous extension to B_2 which is smooth if and only if it is C^1 along a line through the origin 0.

This theorem was proven by K. Jörgens [J] in 1955 in the two-dimensional case. Recently, R. Beyerstedt [B] extended Jörgens' theorem to more general Monge-Ampère equations, also in the case $n = 2$.

We remark that the convexity condition is redundant in the two-dimensional case. In a future work, we intend to show how our multidimensional methods apply to much more general fully nonlinear equations.

1. CONVEXITY

Lemma 2. *Every convex function u in $B_1\backslash\{0\}$ has a convex extension to B_1. It is therefore locally Lipschitz continuous, and $\frac{\partial u}{\partial e}(0)$ exists for any unit vector e. Furthermore,*

$$\frac{\partial u}{\partial(-e)}(0) \geq -\frac{\partial u}{\partial e}(0),$$

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$\frac{\partial u}{\partial e}(te)$ is monotone non-decreasing in t and continuous from the right, i.e.,

$$\lim_{t\to 0^+} \frac{\partial u}{\partial e}(te) = \frac{\partial u}{\partial e}(0).$$

The proof is elementary. The listed properties hold for any $x \in B_1$; and the arguments are simpler, if u is C^1 in $B_1\backslash\{0\}$.

Under this assumption, u is C^1 along the line in the direction e through the origin, iff

$$\frac{\partial u}{\partial(-e)}(0) = -\frac{\partial u}{\partial e}(0).$$

By thickening the line, one can then argue that this is the case iff $\frac{\partial u}{\partial e}$ is C^0 in B_1, because $\frac{\partial u}{\partial e}$ is monotone on all lines in the direction e.

The following theorem is of some interest in its own right. For simplicity, we only consider the two-dimensional case, because this C^1-regularity result is not needed in the proof of Theorem 1.

Proposition 3. *Let u be a convex function in $B_1 \subset \mathbf{R}^2$. Suppose that $\frac{\partial u}{\partial(-e_1)}(0) = -\frac{\partial u}{\partial e_1}(0)$ and that u is C^1 in $B_1\backslash\{0\}$. Then $\frac{\partial u}{\partial e}$ is C^0 in the sector $S_t = \{(x, y) \in B_1 : y \geq t|x|\}$ for any unit vector $e = (a, b)$, $b \geq 0$, and any $t > 0$.*

Proof. First, by subtracting a linear function, we may assume that $u(0, 0) = 0$, $\frac{\partial u}{\partial x}(0, 0) = 0$, $\frac{\partial u}{\partial e_2}(0, 0) = 0$.

By convexity, for $(x, y) \in B_{1/2}$, $y > 0$,

$$u(x, y) \leq \begin{cases} \frac{xu(x+y, 0)+yu(0, x+y)}{x+y} & (x > 0), \\ \frac{xu(x-y, 0)-yu(0, y-x)}{x-y} & (x < 0). \end{cases}$$

Note that the function on the LHS extends $u(x, 0)$ and $u(0, y)$ linearly in the directions $e_1 \mp e_2$. Hence

$$\frac{\partial u}{\partial e}(0, 0) \leq a\frac{\partial u}{\partial e_1}(0, 0) + b\frac{\partial u}{\partial e_2}(0, 0) = 0$$

for all $e = (a, b)$, $b \geq 0$.

On the other hand,

$$u(0, y) \leq \frac{u(x, y) + u(-x, y)}{2}$$

for all $(x, y) \in B_1$, which implies that

$$0 \leq 2b\frac{\partial u}{\partial e_2}(0, 0) \leq \frac{\partial u}{\partial e}(0, 0) + \frac{\partial u}{\partial e^-}(0, 0),$$

where $e^- = (-a, b)$. Therefore

$$\frac{\partial u}{\partial e}(0, 0) = 0$$

for all $e = (a, b)$, $b \geq 0$. It follows that

$$\frac{\partial u}{\partial(-e)}(0, 0) \geq 0,$$

and in turn that $u \geq 0$ in B_1.

For $t > 0$, consider the direction $e_t = (a, b)$, $b = ta > 0$. Then, by the monotonicity of $\frac{\partial u}{\partial e_t}$ on all lines in the direction e_t, for any given $\varepsilon < 0$, there is a $\delta > 0$ such that

$$\left|\frac{\partial u}{\partial e_t}\right| \leq \varepsilon$$

in the parallelogram $\{(x, y) \in B_1 : 0 \leq x \leq a,\ tx \leq y \leq tx + \delta\}$.

A similar argument can be made for e_t^-, which implies that $\frac{\partial u}{\partial e}$ is C^0 in S_t, for any $e = (a, b)$, $b \geq 0$, as required. □

2. The comparison principle

We show that u satisfies the comparison principle. In this section, we use the notation $B_r(x) = x + B_r$.

Lemma 4. *Let v_1 and v_2 be C^2 in $B_{1/2}(\frac{1}{2}e_1)$ and C^0 in $\overline{B_{1/2}(\frac{1}{2}e_1)}$, which solve the equation*

$$\det D^2 u = 1 \quad in\ B_{1/2}(\tfrac{1}{2}e_1).$$

Assume further that $v_1(0) = v_2(0)$, $v_2 \geq v_1$ on $\partial B_{1/2}(\frac{1}{2}e_1)$, and $v_1 \not\equiv v_2$. Then

(a) $v_2 \geq v_1$ *in* $B_{1/2}(\frac{1}{2}e_1)$.

(b) *If we assume further that v_1 and v_2 are C^2 in $\overline{B_{1/2}(\frac{1}{2}e_1)}$, then*

$$\frac{\partial v_2}{\partial x_1}(0) > \frac{\partial v_1}{\partial x_1}(0).$$

Proof. (a) follows from the weak maximum principle [GT], Theorem 17.1, page 443, which in turn follows from the classical weak maximum principle [GT], Theorem 3.1.

To prove (b), let $w = v_2 - v_1$. Then

$$\sum_{i,j=1}^{n} A_{ij}(x)\frac{\partial^2 w}{\partial x_i \partial x_j} = \det D^2 v_2 - \det D^2 v_1 = 0,$$

where $A_{ij}(x) = \partial \det(\theta D^2 v_2 + (1-\theta)D^2 v_1)/\partial u_{ij}$ for some $\theta = \theta(x)$, $0 < \theta < 1$, which depends on $D^2 v_1$ and $D^2 v_2$. Since $[A_{ij}]$ is uniformly elliptic, the classical strong maximum principle [GT], Theorem 3.5, together with the Hopf lemma [GT], Lemma 3.4, yield statement (b). □

Lemma 5. *Let v_1 and v_2 be as in Lemma 4. Suppose further that v_2 is C^3 in $\overline{B_{1/2}(\frac{1}{2}e_1)}$ and that $v_2(e_1) > v_1(e_1)$. Then $\frac{\partial v_2}{\partial x_1}(0) > \frac{\partial v_1}{\partial x_1}(0)$.*

Proof. Let φ_2 be a smooth function such that $\varphi_2 = v_2$ on $\partial B_{1/2}(\frac{1}{2}e_1) \cap B_{1/2}(0)$, $v_1 < \varphi_2 < v_2$ near e_1, and $v_1 \leq \varphi_2 \leq v_2$ on $\partial B_{1/2}(\frac{1}{2}e_1)$. Let u_2 be the solution of

$$\begin{cases} \det D^2 u_2 = 1 & \text{in } B_{1/2}(\frac{1}{2}e_1), \\ u_2 = \varphi_2 & \text{on } \partial B_{1/2}(\frac{1}{2}e_1), \end{cases}$$

whose existence follows from [GT], Theorem 17.22, page 473, in combination with Problem 17.11 (ii), page 490. By Lemma 5, $v_2 \geq u_2 \geq v_1$ in $B_{1/2}(\frac{1}{2}e_1)$.

Moreover,

$$\frac{\partial v_2}{\partial x_1}(0) > \frac{\partial u_2}{\partial x_1}(0).$$

Hence

$$\frac{\partial v_2}{\partial x_1}(0) > \frac{\partial u_2}{\partial x_1}(0) \geq \frac{\partial v_1}{\partial x_1}(0). \quad \square$$

Proof of the main theorem. Let v be the solution of

$$\begin{cases} \det D^2 v = 1 & \text{in } B_1\,, \\ v = u & \text{on } \partial B_1(0). \end{cases}$$

The claim is that $u = v$. We only show $v \geq u$ by contradiction, since the other part is similar.

Let ε be the minimum such that

$$v \geq u - \varepsilon \quad \text{in } \overline{B_1(0)}.$$

By the assumption, $\varepsilon > 0$ and $v(x_0) = u(x_0) - \varepsilon$ for some $x_0 \in \overline{B_1(0)}$.

We may assume that $x_0 = 0$, because otherwise the classical strong maximum principle implies that $v \equiv u - \varepsilon$, which is a contradiction. If $x_0 = 0$, then $\frac{\partial v}{\partial x_1}(0) = \frac{\partial u}{\partial x_1}(0)$, since we may assume that u is C^1 along the x_1-axis.

However, Lemma 5 now implies that

$$\frac{\partial v}{\partial x_1}(0) > \frac{\partial u}{\partial x_1}(0),$$

which is a contradiction. $\square$

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