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OSTROWSKI TYPE INEQUALITIES

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ABSTRACT. Optimal upper bounds are given to the deviation of a function $f \in C^N([a,b])$, $N \in \mathbb{N}$, from its averages. These bounds are of the form $A \cdot \|f^{(N)}\|_\infty$, where A is the smallest universal constant, i.e., the produced inequalities are sharp and sometimes are attained. This work has been greatly motivated by the works of Ostrowski (1938) and Fink (1992).

0. INTRODUCTION

Here we establish optimal upper bounds on the deviation of a function from its averages. These lead to sharp inequalities. Namely, let $f \in C^{n+1}([a,b])$, $n \in \mathbb{Z}_+$, such that $f^{(k)}(x) = 0$, $k = 1, \ldots, n$, where x is a fixed point in $[a,b]$. Then we establish that

$$(*) \qquad \left| \frac{1}{b-a} \cdot \int_a^b f(y)\,dy - f(x) \right| \le \varphi_n(x) \cdot \|f^{(n+1)}\|_\infty ,$$

where $\varphi_n(x)$ is a continuous function that depends only on n, a, b, it has a simple form and it is the smallest possible, i.e., $(*)$ is sharp and in some cases it is even attained. The special case of $x = \frac{a+b}{2}$ is encountered.

1. ON OSTROWSKI'S INEQUALITY

Ostrowski's inequality (see Ostrowski [2]) is as follows:

$$(1) \qquad \left| \frac{1}{b-a} \cdot \int_a^b f(y)dy - f(x) \right| \le \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right) \cdot (b-a) \cdot \|f'\|_\infty ,$$

where $f \in C^1([a,b])$, $x \in [a,b]$. Inequality (1) is sharp since the function in ( ) cannot be replaced by a smaller one. One can easily notice that

$$(2) \qquad \left(\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right) \cdot (b-a) = \frac{(x-a)^2 + (b-x)^2}{2 \cdot (b-a)}.$$

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Next, we give a different proof to (1) from that of Ostrowski's initial proof of 1938 in [2].

Theorem 1. *Let* $f \in C^1([a, b])$, $x \in [a, b]$. *Then*

$$\left|\frac{1}{b-a} \cdot \int_a^b f(y)dy - f(x)\right| \leq \left(\frac{(x-a)^2 + (b-x)^2}{2 \cdot (b-a)}\right) \cdot \|f'\|_\infty. \tag{3}$$

Inequality (3) *is sharp, namely the optimal function is*

$$f^*(y) := |y-x|^\alpha \cdot (b-a), \qquad \alpha > 1. \tag{4}$$

Proof. Observe that

$$\begin{aligned}
\left|\frac{1}{b-a} \cdot \int_a^b f(y)dy - f(x)\right| &= \frac{1}{(b-a)} \cdot \left|\int_a^b (f(y) - f(x))dy\right| \\
&\leq \frac{1}{(b-a)} \cdot \int_a^b |f(y) - f(x)| \cdot dy \\
&\leq \frac{1}{(b-a)} \cdot \|f'\|_\infty \cdot \int_a^b |y-x| \cdot dy \\
&= \frac{\|f'\|_\infty}{2 \cdot (b-a)} \cdot ((x-a)^2 + (b-x)^2).
\end{aligned}$$

So, we have established inequality (3). Note that

$$f^{*\prime}(y) = \alpha \cdot |y-x|^{\alpha-1} \cdot \operatorname{sign}(y-x) \cdot (b-a),$$

thus

$$|f^{*\prime}(y)| = \alpha \cdot |y-x|^{\alpha-1} \cdot (b-a)$$

and

$$\|f^{*\prime}\|_\infty = \alpha \cdot (b-a) \cdot (\max(b-x, x-a))^{\alpha-1}.$$

Also we notice that $f^*(x) = 0$.

Therefore we have for f^* that

$$\text{L.H.S.}(3) = \int_a^b |y-x|^\alpha \cdot dy = \frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{\alpha+1}$$

and

$$\lim_{\alpha \to 1} \text{L.H.S.}(3) = \frac{(x-a)^2 + (b-x)^2}{2}. \tag{5}$$

Also, we observe that

$$\text{R.H.S.}(3) = \left(\frac{(x-a)^2 + (b-x)^2}{2}\right) \cdot \alpha \cdot (\max(b-x, x-a))^{\alpha-1}$$

and

$$\lim_{\alpha \to 1} \text{R.H.S.}(3) = \frac{(x-a)^2 + (b-x)^2}{2},$$

i.e.,

$$\lim_{\alpha \to 1} \text{L.H.S.}(3) = \lim_{\alpha \to 1} \text{R.H.S.}(3),$$

proving (3) sharp. □

Note that when $x = a$ or $x = b$, inequality (3) can be attained by $f_a(y) := (y-a)\cdot(b-a)$, $f_b(y) := (y-b)\cdot(b-a)$, respectively (then both sides of (3) are equal to $(b-a)^2/2$).

2. More general Ostrowski type inequalities

The following material has been greatly motivated by the important work of Fink [1]. Let $f \in C^{n+1}([a,b])$, $n \in \mathbb{N}$, $x \in [a,b]$, be fixed. Then by Taylor's theorem we get

$$f(y) - f(x) = \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}\cdot(y-x)^k + \mathcal{R}_n(x,y), \tag{6}$$

where

$$\mathcal{R}_n(x,y) := \int_x^y (f^{(n)}(t) - f^{(n)}(x))\cdot\frac{(y-t)^{n-1}}{(n-1)!}\cdot dt; \tag{7}$$

here y can be $\geq x$ or $\leq x$.

Let $y \geq x$; then

$$\begin{aligned}
|\mathcal{R}_n(x,y)| &\leq \int_x^y |f^{(n)}(t) - f^{(n)}(x)|\cdot\frac{(y-t)^{n-1}}{(n-1)!}\cdot dt \\
&\leq \|f^{(n+1)}\|_\infty \cdot \int_x^y |t-x|\cdot\frac{|y-t|^{n-1}}{(n-1)!}\cdot dt \\
&= \|f^{(n+1)}\|_\infty \cdot \frac{(y-x)^{n+1}}{(n+1)!},
\end{aligned}$$

i.e.,

$$|\mathcal{R}_n(x,y)| \leq \frac{\|f^{(n+1)}\|_\infty}{(n+1)!}\cdot(y-x)^{n+1}, \qquad y \geq x. \tag{8}$$

Now let $x \geq y$; then

$$\begin{aligned}
|\mathcal{R}_n(x,y)| &= \left|\int_y^x (f^{(n)}(t) - f^{(n)}(x))\cdot\frac{(y-t)^{n-1}}{(n-1)!}\cdot dt\right| \\
&\leq \int_y^x |f^{(n)}(t) - f^{(n)}(x)|\cdot\frac{|y-t|^{n-1}}{(n-1)!}\cdot dt \\
&\leq \frac{\|f^{(n+1)}\|_\infty}{(n-1)!}\cdot\int_y^x (x-t)\cdot(t-y)^{n-1}\cdot dt \\
&= \frac{\|f^{(n+1)}\|_\infty}{(n+1)!}\cdot(x-y)^{n+1},
\end{aligned}$$

i.e.,

$$|\mathcal{R}_n(x,y)| \leq \frac{\|f^{(n+1)}\|_\infty}{(n+1)!}\cdot(x-y)^{n+1}, \qquad x \geq y. \tag{9}$$

From (8) and (9) we get

$$|\mathcal{R}_n(x,y)| \leq \frac{\|f^{(n+1)}\|_\infty}{(n+1)!}\cdot|y-x|^{n+1}, \quad \text{for all } x, y \in [a,b]. \tag{10}$$

Next we treat

$$\begin{aligned}
\left|\frac{1}{b-a}\cdot\int_a^b f(y)dy - f(x)\right| &= \frac{1}{b-a}\cdot\left|\int_a^b (f(y)-f(x))\cdot dy\right| \\
&= \frac{1}{b-a}\cdot\left|\int_a^b \left[\sum_{k=1}^n \frac{f^{(k)}(x)}{k!}\cdot(y-x)^k + \mathscr{R}_n(x,y)\right]\cdot dy\right| \\
&= \frac{1}{b-a}\cdot\left|\sum_{k=1}^n \frac{f^{(k)}(x)}{k!}\cdot\int_a^b (y-x)^k\cdot dy + \int_a^b \mathscr{R}_n(x,y)\cdot dy\right| \\
&= \frac{1}{b-a}\cdot\left|\sum_{k=1}^n \frac{f^{(k)}(x)}{(k+1)!}\cdot[(b-x)^{k+1}-(a-x)^{k+1}]\right. \\
&\qquad\left. + \int_a^b \mathscr{R}_n(x,y)\cdot dy\right| \quad \text{(by (10))} \\
&\le \frac{1}{b-a}\cdot\left[\sum_{k=1}^n \frac{|f^{(k)}(x)|}{(k+1)!}\cdot|(b-x)^{k+1}-(a-x)^{k+1}|\right. \\
&\qquad\left. + \frac{\|f^{(n+1)}\|_\infty}{(n+1)!}\cdot\int_a^b |y-x|^{n+1}\cdot dy\right],
\end{aligned}$$

i.e., we have proved that

$$\begin{aligned}
&\left|\frac{1}{b-a}\cdot\int_a^b f(y)dy - f(x)\right| \\
&\le \frac{1}{b-a}\cdot\left[\sum_{k=1}^n \frac{|f^{(k)}(x)|}{(k+1)!}\cdot|(b-x)^{k+1}-(a-x)^{k+1}|\right. \\
&\qquad\left. + \frac{\|f^{(n+1)}\|_\infty}{(n+2)!}\cdot((x-a)^{n+2}+(b-x)^{n+2})\right],
\end{aligned} \tag{11}$$

where $f\in C^{n+1}([a,b])$, $n\in\mathbb{N}$, $x\in[a,b]$, is fixed.

If we choose $x=\frac{a+b}{2}$, then

$$b-x=x-a=\frac{b-a}{2}.$$

Thus

(12)

$$\begin{aligned}
&\left|\frac{1}{b-a}\cdot\int_a^b f(y)dy - f\left(\frac{a+b}{2}\right)\right| \\
&\le \frac{1}{b-a}\cdot\left[\sum_{1\le k\ \text{even}\le n} \frac{|f^{(k)}(\frac{a+b}{2})|}{(k+1)!}\cdot\frac{(b-a)^{k+1}}{2^k} + \frac{\|f^{(n+1)}\|_\infty}{(n+2)!}\cdot\frac{(b-a)^{n+2}}{2^{n+1}}\right],
\end{aligned}$$

where $f\in C^{n+1}([a,b])$, $n\in\mathbb{N}$.

The above considerations and the established inequalities (11) and (12) lead to the following results.

Theorem 2. *Let $f \in C^{n+1}([a, b])$, $n \in \mathbb{N}$ and $x \in [a, b]$ be fixed, such that $f^{(k)}(x) = 0$, $k = 1, \ldots, n$. Then*

$$\left|\frac{1}{b-a} \cdot \int_a^b f(y)dy - f(x)\right| \leq \frac{\|f^{(n+1)}\|_\infty}{(n+2)!} \cdot \left(\frac{(x-a)^{n+2} + (b-x)^{n+2}}{b-a}\right). \tag{13}$$

Inequality (13) *is sharp. Namely, when n is odd it is attained by $f^*(y) := (y-x)^{n+1} \cdot (b-a)$, while when n is even the optimal function is*

$$\tilde{f}(y) := |y-x|^{n+\alpha} \cdot (b-a), \qquad \alpha > 1.$$

Proof. Inequality (13) comes immediately from (11). Next we prove the sharpness of inequality (13).

When n is odd: Notice that $f^{*(k)}(x) = 0, k = 0, 1, \ldots, n$, and $f^{*(n+1)}(y) = (n+1)! \cdot (b-a)$. Hence

$$\|f^{*(n+1)}\|_\infty = (n+1)! \cdot (b-a).$$

Plugging f^* into (13) we get

$$\text{L.H.S.}(13) = \frac{(b-x)^{n+2} + (x-a)^{n+2}}{n+2}. \tag{14}$$

Also,

$$\text{R.H.S.}(13) = \frac{(x-a)^{n+2} + (b-x)^{n+2}}{n+2}. \tag{15}$$

From (14) and (15), when n is odd, inequality (13) was proved to be sharp, in particular attained by f^*.

When n is even: Notice that $\tilde{f}^{(k)}(x) = 0$, $k = 0, 1, \ldots, n$, and

$$\tilde{f}^{(n+1)}(y) = (n+\alpha)(n+\alpha-1)\cdots(\alpha+1) \cdot \alpha \cdot |y-x|^{\alpha-1} \cdot \operatorname{sign}(y-x) \cdot (b-a).$$

Hence

$$|\tilde{f}^{(n+1)}(y)| = \left(\prod_{j=0}^{n}(n+\alpha-j)\right) \cdot |y-x|^{\alpha-1} \cdot (b-a)$$

and

$$\|\tilde{f}^{(n+1)}\|_\infty = \left(\prod_{j=0}^{n}(n+\alpha-j)\right) \cdot (\max(b-x, x-a))^{\alpha-1} \cdot (b-a).$$

Consequently we have

$$\text{R.H.S.}(13) = \frac{(\prod_{j=0}^{n}(n+\alpha-j)) \cdot (\max(b-x, x-a))^{\alpha-1}}{(n+2)!} \cdot ((x-a)^{n+2} + (b-x)^{n+2}), \qquad \alpha > 1.$$

Thus

$$\lim_{\alpha-1} \text{R.H.S.}(13) = \frac{(x-a)^{n+2} + (b-x)^{n+2}}{n+2} \tag{16}$$

and

$$\text{L.H.S.}(13) = \frac{(x-a)^{n+\alpha+1} + (b-x)^{n+\alpha+1}}{n+\alpha+1}.$$

Therefore

$$\lim_{\alpha\to 1} \text{L.H.S.}(13) = \frac{(x-a)^{n+2}+(b-x)^{n+2}}{n+2}. \tag{17}$$

From (16) and (17) we get that (13) is sharp also when n is even. □

Note that when $x = a$ or $x = b$ and n is even, inequality (13) can be attained by $\tilde{f}_a(y) := (y-a)^{n+1}\cdot(b-a)$, $\tilde{f}_b(y) := (y-b)^{n+1}\cdot(b-a)$, respectively (then both sides of (13) are equal to $(b-a)^{n+2}/n+2)$. When $x = (a+b)/2$, we have a case of special interest which is described next.

Theorem 3. *Let* $f \in C^{n+1}([a, b])$, $n \in \mathbb{N}$ *such that* $f^{(k)}((a+b)/2) = 0$, *all* k *even* $\in \{1, \dots, n\}$. *Then*

$$\left|\frac{1}{b-a}\cdot\int_a^b f(y)dy - f\left(\frac{a+b}{2}\right)\right| \le \frac{\|f^{(n+1)}\|_\infty}{(n+2)!}\cdot\frac{(b-a)^{n+1}}{2^{n+1}}. \tag{18}$$

Inequality (18) *is sharp. Namely, when* n *is odd it is attained by* $f^*(y) := (y-\frac{a+b}{2})^{n+1}\cdot(b-a)$, *while when* n *is even the optimal function is*

$$\tilde{f}(y) := \left|y-\frac{a+b}{2}\right|^{n+\alpha}\cdot(b-a), \qquad \alpha > 1.$$

Corollary 1. *Let* $f \in C^2([a, b])$ *such that* $f''((a+b)/2) = 0$. *Then*

$$\left|\frac{1}{b-a}\cdot\int_a^b f(y)dy - f\left(\frac{a+b}{2}\right)\right| \le \|f''\|_\infty\cdot\frac{(b-a)^2}{24}, \tag{19}$$

which is sharp as in Theorem 3.

Proof. Apply Theorem 3 with $n = 1$.

Proof of Theorem 3. Inequality (18) comes immediately from (12) and the assumption $f^{(k)}((a+b)/2) = 0$, all k even in $\{1, \dots, n\}$.

Next we prove the sharpness of inequality (18).

When n is odd: We notice that

$$f^{*(k)}\left(\frac{a+b}{2}\right) = 0, \quad \text{for } k = 0 \text{ and all } k \text{ even } \in \{1, \dots, n\},$$

and furthermore

$$f^{*(n+1)}(y) = (n+1)!\cdot(b-a), \quad \text{all } y \in [a, b].$$

Thus

$$\text{R.H.S.}(18) = \frac{(b-a)^{n+2}}{(n+2)\cdot 2^{n+1}}. \tag{20}$$

Also we find

$$\text{L.H.S.}(18) = \frac{(b-a)^{n+2}}{(n+2)\cdot 2^{n+1}}. \tag{21}$$

From (20) and (21) we get that (18) is attained by f^*, therefore (18) has been proved as sharp when n is odd.

When n is even: We notice that furthermore $\tilde{f}^{(k)}((a+b)/2) = 0$, for $k = 0$ and all k even in $\{1, \dots, n\}$,

$$\tilde{f}^{(n+1)}(y) = \prod_{j=0}^{n}(n+\alpha-j) \cdot \left|y - \frac{a+b}{2}\right|^{\alpha-1} \cdot \operatorname{sign}\left(y - \frac{a+b}{2}\right) \cdot (b-a)$$

and

$$\|\tilde{f}^{(n+1)}\|_{\infty} = \left(\prod_{j=0}^{n}(n+\alpha-j)\right) \cdot \left(\frac{b-a}{2}\right)^{\alpha-1} \cdot (b-a).$$

Thus

$$\begin{aligned} \text{R.H.S.}(18) &= \frac{(\prod_{j=0}^{n}(n+\alpha-j)) \cdot ((b-a)/2)^{\alpha-1} \cdot (b-a)}{(n+2)!} \\ &\quad \cdot \frac{(b-a)^{n+1}}{2^{n+1}}, \qquad \alpha > 1. \end{aligned}$$

Hence

$$\lim_{\alpha \to 1} \text{R.H.S.}(18) = \frac{(b-a)^{n+2}}{(n+2) \cdot 2^{n+1}}. \tag{22}$$

Also we find

$$\text{L.H.S.}(18) = \frac{2 \cdot ((b-a)/2)^{n+\alpha+1}}{n+\alpha+1}$$

and

$$\lim_{\alpha \to 1} \text{L.H.S.}(18) = \frac{(b-a)^{n+2}}{2^{n+1} \cdot (n+2)}. \tag{23}$$

From (22) and (23) we have established that inequality (18) is sharp again when n is even. □

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