

HARDY-BOHR POSITIVITY

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ABSTRACT. We consider two general principles that lead to Hardy-Bohr positivity. These are applied to give a simple proof that the Cesàro methods of positive order have Hardy-Bohr positivity.

1. INTRODUCTION

For matrix methods of summability A , where A is a regular matrix triangle, determining the summability factors has attracted the attention of many authors. In the theory of sequence spaces the notion of a sum space and the so-called Hardy-Bohr positivity play important roles. We consider two general principles that lead to Hardy-Bohr positivity. In section 3 we show that any triangle with a diapositive inverse has Hardy-Bohr positivity and apply this to show that the Cesàro methods (C, α) , for $0 \leq \alpha \leq 1$, have this property. In section 4 we consider convolution triangles and show that the product of two convolution triangles with Hardy-Bohr positivity also has Hardy-Bohr positivity. This result together with the result of section 3 gives Hardy-Bohr positivity for the Cesàro methods of all orders $\alpha \geq 0$. The proof for the Cesàro methods is significantly easier than any previously given. In section 5 we conclude with several comments on Nörlund methods, which are not Cesàro-like and have Hardy-Bohr positivity.

2. NOTATION AND TERMINOLOGY

Throughout we use notation and results given by Wilansky [20] and by Zeller-Beekmann [21]. Let ω denote the space of all sequences, m the space of bounded sequences, c the convergent sequences, c_0 sequences that converge to 0, $cs = \{x: \sum_n x_n \text{ is convergent}\}$, $l_1 = \{x: \sum_n |x_n| < \infty\}$, and φ all finitely nonzero sequences. If $A = (a_{nk})$ is an infinite matrix, the matrix method A defines a sequence-to-sequence transformation, mapping a sequence s (real or complex terms) to t

$$t_n = (Ax)_n = \sum_{k=0}^{\infty} a_{nk} s_k, \quad n = 0, 1, 2, \dots$$

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The convergence domain c_A for the matrix method A consists of those sequences s for which $t = As$ exists and belongs to c . The A -limit is defined for $s \in c_A$ by $A - \lim s_k = \lim t_n$. The method A is conservative provided $c \subset c_A$ and strictly stronger than convergence if, moreover, $c \neq c_A$. A conservative method is regular provided $A - \lim s_k = \lim s_k$ for all $s \in c$. The matrix A is a triangle provided $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all n . Let Σ denote the triangle of ones, so that $A\Sigma$ is the series-to-sequence method associated with A .

If E is an FK -space containing φ , the multipliers on E are defined as $M(E) = \{x \in \omega: xy \in E \forall x \in E\}$, where xy denotes the coordinatewise product. The f -dual of E is $E^f = \{(f(e_i))_{i=1}^\infty: f \in E'\}$ where E' is the topological dual of E and e_i denotes the i th coordinate sequence. Then E is said to be a sum space provided $E^f = M(E)$. The notion of a sum space was defined and studied by Ruckle in [16], [17], [18], and [19]. For example, let B be a c -reversible, row-finite, sp_1 matrix (i.e., column limits are all one) such that $l_1 \subseteq c_B$ or c_B has AD (i.e., φ is dense). Then c_B is a sum space if and only if c_B has B -sectional boundedness if and only if c_B has B -sectional convergence [16], [8]. In particular, if A is a regular triangle, then $B = A\Sigma$ is a c -reversible, row-finite, sp_1 matrix such that $l_1 \subseteq c_B$. In [22], Zeller showed that if $B = (C, \alpha)\Sigma$, $\alpha \geq 0$, then c_B has B -sectional convergence and hence the series-to-sequence Cesàro methods of order $\alpha \geq 0$ are sum spaces. If a sequence β has the property that $\sum_n \beta_n s_n$ is summable B whenever $\sum_n s_n$ is summable A we say β is a summability factor of type (A, B) and write $\beta \in (A, B)$. For a triangle $A = (a_{nk})$ (sequence-to-sequence method) let $A\Sigma = (\bar{a}_{nk})$ be the corresponding series-to-sequence method where $\bar{a}_{nk} = \sum_{j=k}^n a_{nj}$ if $k \leq n$ and $\bar{a}_{nk} = 0$ if $k > n$. An ordered pair of triangles $[A, B]$ belongs to the class \mathcal{L} , or satisfies the Hardy-Bohr criteria, if the conditions

$$\beta_n = d + \sum_{j=n}^{\infty} \bar{a}_{jn} \gamma_j,$$

$$\beta_n = O(a_{nn}/b_{nn})$$

where d is a constant and $\gamma = (\gamma_j) \in l_1$ (d and γ depend on β) are necessary and sufficient for $\beta \in (A, B)$. It follows from [15], [7], [13], and [14] that $[(C, \alpha), (C, \beta)] \in \mathcal{L}$, $\forall \alpha, \beta \geq 0$. Many authors have considered the problem of identifying pairs of methods in the class \mathcal{L} . In [9] the observation is made that for a regular triangle A , $[A, A] \in \mathcal{L}$ if and only if $c_{A\Sigma}$ is a sum space. Moreover, the regular triangular methods A for which $[A, A] \in \mathcal{L}$ are precisely those methods for which the summability factors represent the continuous linear functionals on $c_{A\Sigma}$. A regular triangle A is said to have Hardy-Bohr positivity, or HB-positivity, provided

$$\sum_{l=j}^k (a_{ij}^{-1} - a_{i-1,j}^{-1}) \bar{a}_{nl} \bar{a}_{kl} \geq 0, \quad \forall 0 \leq j \leq k \leq n,$$

where $A^{-1} = (a_{nk}^{-1})$. If A has HB-positivity, then the series-to-sequence convergence domain $c_{A\Sigma}$ is a sum space [9]. Let Σ^α , $\alpha \geq 0$, be the convolution method generated by $p(x) = 1/(1-x)^\alpha$. In [12] it is shown that Σ^α has HB-positivity, and from this one easily argues the HB-positivity of (C, α) . In this

article, we establish the HB-positivity of Σ^α for $0 \leq \alpha \leq 1$ using Theorem 1, whereas, the HB-positivity of Σ^α for $\alpha > 1$ then follows from Theorem 2. See [12], [1], [10], [2], [3], [4], [5], [6], and [21].

For a triangle A and $B = A\Sigma$ we let D_r denote the diagonal matrix with the r th row of B on the diagonal. That is $d_{kk} = b_{rk}$ for all $k \geq 0$. Then A having HB-positivity is equivalent to

$$BD_r B^{-1} \geq 0, \quad \forall r.$$

In the next two sections we give two general principles leading to HB-positivity together covering the Cesàro methods and yielding a simple proof.

3. DIAPOSITIVE INVERSE

A triangle $A = (a_{nk})$ is said to be diapositive provided $a_{nn} > 0$ for all $n \geq 0$ and $a_{nk} \leq 0$ for $n \neq k$. See [21] and [23].

Theorem 1. *Let A be a regular triangle and $B = A\Sigma$. If A^{-1} is diapositive, then A has HB-positivity.*

Proof. First we observe that $A \geq 0$ since A^{-1} is diapositive (use geometric series or recursion). Hence in each row of B the elements decrease to 0 (and are eventually all equal to 0). Next we consider $BD_r B^{-1} A = BD_r \Sigma^{-1}$. Then the elements of row m of this matrix are given by

$$b_{m,0}b_{r,0} - b_{m,1}b_{r,1}, b_{m,1}b_{r,1} - b_{m,2}b_{r,2}, \dots,$$

and since $B\Sigma^{-1} = A$, the elements in row m are given by

$$a_{r,0}b_{m,0} + a_{m,0}b_{r,1}, a_{r,1}b_{m,1} + a_{m,1}b_{r,2}, \dots$$

The terms $a_{r,k}b_{m,k}$ for $r, k \geq 0$ constitute the matrix AD_m . Multiplying by A^{-1} we are led to

$$AD_m A^{-1} \geq 0$$

where the inequality is true for the following reasons. The elements of row r of this latter matrix are of the form

$$a_{r,0}a_{0,0}^{-1}b_{m,0} + \dots + a_{r,r}a_{r,0}^{-1}b_{m,r}, a_{r,1}a_{1,1}^{-1}b_{m,1} + \dots + a_{r,r}a_{r,1}^{-1}b_{m,r}, \dots$$

In $AA^{-1} = I \geq 0$, modified by the terms of B , the influence of the negative terms is diminished in comparison to the positive term when multiplied by the decreasing factors of D_m . Similarly, the terms $a_{m,k}b_{r,k+1}$ constitute the matrix A times D_m shifted and when multiplied on the right by A^{-1} the nonnegativity follows as above. \square

As an application of Theorem 1, since Σ^α for $0 \leq \alpha \leq 1$ has a diapositive inverse, the methods have HB-positivity and hence we have a simple proof that the Cesàro methods of order α , $0 \leq \alpha \leq 1$, also have HB-positivity.

4. CONVOLUTION TRIANGLES

Given a suitable sequence $p = (p_k)$ the triangle $A = (a_{nk}) = (p_{n-k})$, $n, k \geq 0$, is called a convolution triangle (or Toeplitz matrix) [11]. The product of two convolution triangles is another convolution triangle generated by the Cauchy

product of the two original generating sequences. For the convolution triangle A define the function p by

$$p(x) = \sum_{k=0}^{\infty} p_k x^k.$$

Let

$$P(x) = \frac{p(x)}{(1-x)} = \sum_{k=0}^{\infty} P_k x^k,$$

where $P_k = \sum_{k=0}^n p_k$. Such a convolution triangle A has HB-positivity if and only if the coefficients of the power series

$$\frac{P(x)P(y)}{P(xy)}$$

are nonnegative. That is the series is absolutely monotone [1]. Let

$$A(x) = \frac{1-x}{p(x)}.$$

Then HB-positivity follows immediately from

$$\begin{aligned} Q &= \frac{P(x)P(y)}{P(xy)} = \sum_{i,j,k=0}^{\infty} P_j P_k A_i (xy)^i x^k y^j \\ &= \sum_{i,j,k=0}^{\infty} P_j P_k A_i x^{i+k} y^{i+j}, \end{aligned}$$

and letting $m = i + k$, $n = i + j$,

$$Q = \sum_{m,n=0}^{\infty} \left\{ \sum_{i=0}^{\min(m,n)} P_{m-i} P_{n-i} (a_i - a_{i-1}) \right\} x^m y^n$$

where $a(x) = 1/p(x)$. This is precisely the condition for HB-positivity [12].

Theorem 2. *If S and T are convolution triangles with HB-positivity, then ST has HB-positivity.*

Proof. If the functions for S and T are p and q respectively, then the function associated with ST is simply pq . The criterion together with the commutativity of P and Q leads to the condition that

$$\frac{P(x)P(y)}{P(xy)} \frac{Q(x)Q(y)}{Q(xy)}$$

is absolutely monotone. \square

As an application of Theorem 2 we have Σ^α for $\alpha \geq 1$ has HB-positivity and hence as before the Cesàro methods of order $\alpha \geq 1$ have HB-positivity. This follows from Theorem 1 and Theorem 2 and transfinite induction.

5. REMARKS

The Nörlund methods provide additional interesting non-Cesàro-like triangles with HB-positivity. In [10], the Nörlund method N_p generated by $p(x) =$

$(1+x)/(1-x)$ is shown to have HB-positivity, so that $c_{N_p\Sigma}$ is a sum space and, moreover, $[N_p, N_p] \in \mathcal{L}$ so that the summability factors are given by the classical conditions. As in the case of Σ^α and (C, α) , if the convolution triangle generated by the function $p(x)$ has HB-positivity, then the Nörlund mean generated by $p(x)$ will also have HB-positivity. Arguing as in Theorem 2, if a convolution triangle generated by $p(x)$ has HB-positivity, then the convolution triangle generated by $p(x)/(1-x)^\alpha$, $\alpha \geq 0$, will also have HB-positivity. In particular, the Nörlund methods generated by $(1+x)/(1-x)^\alpha$, $\alpha \geq 1$, have HB-positivity. In a similar manner, if M and N are positive integers with $N \geq 2M - 1$, then the Nörlund method generated by

$$p(x) = (1+x)^M/(1-x)^N$$

has HB-positivity. Computer analysis suggests this latter result is best possible.

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