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FINITELY GRADED LOCAL COHOMOLOGY AND THE DEPTHS OF GRADED ALGEBRAS

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ABSTRACT. The term "finitely graded" is introduced here to refer to graded modules which are nonzero in only finitely many graded pieces. We consider the question of when the local cohomology modules of a graded module are finitely graded. Using a theorem of Faltings concerning the annihilation of local cohomology, we obtain some partial answers to this question. These results are then used to compare the depths of the Rees algebra and the associated graded ring of an ideal in a local ring.

1. INTRODUCTION

Let R be a commutative local Noetherian ring of positive Krull dimension and let I be an ideal of R. Two important graded rings which can be formed from R and I are the *Rees algebra* of I, $R[It] = \bigoplus_{n\geq 0} I^n t^n$, and the *associated graded ring* of I, $gr_I(R) = \bigoplus_{n\geq 0} I^n/I^{n+1}$. In recent years there has been considerable interest in the question of when $gr_I(R)$ being Cohen-Macaulay implies that $R[It]$ is Cohen-Macaulay. Since $\dim R[It] = \dim R + 1 = \dim gr_I(R) + 1$ (provided $\dim R/I < \dim R$; see [V]), this question can be rephrased in the following way: if $gr_I(R)$ is Cohen-Macaulay, when is $\operatorname{depth}_m R[It] = \operatorname{depth}_m gr_I(R) + 1$, where m is the homogeneous maximal ideal of $R[It]$? Answers to this question have been given in [AHT], [GH], [GS], [JK], [SUV], and [TI]. All of these papers indicate that if $gr_I(R)$ is Cohen-Macaulay, then some additional assumptions on the ideal I are needed to conclude $\operatorname{depth}_m R[It] = \operatorname{depth}_m gr_I(R) + 1$. However, S. Huckaba and the present author ([HM1], [HM2]) show that $\operatorname{depth}_m R[It] = \operatorname{depth}_m gr_I(R) + 1$ whenever $\operatorname{depth}_m gr_I(R) < \operatorname{depth} R$. In particular, if R is Cohen-Macaulay and $gr_I(R)$ is not Cohen-Macaulay, then

$$\operatorname{depth}_m R[It] = \operatorname{depth}_m gr_I(R) + 1.$$

In this paper, we show that a similar result (Theorem 3.4) holds for the depth with respect to any homogeneous ideal containing $IR[It]$.

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Theorem 3.4. *Let J be a homogeneous ideal of $R[It]$ which contains $IR[It]$. Then*

(a) $\operatorname{depth}_J gr_I(R) \leq \operatorname{depth}_J R[It]$,
(b) $\operatorname{depth}_J gr_I(R) \leq \operatorname{depth}_{J\cap R} R$, *and*
(c) *if* $\operatorname{depth}_J gr_I(R) < \operatorname{depth}_{J\cap R} R$, *then* $\operatorname{depth}_J R[It] = \operatorname{depth}_J gr_I(R)+1$.

The proof of this theorem relies on the intricate relationship between the local cohomology of $R[It]$, $gr_I(R)$ and R. A key step in the proof is determining for which integers i the modules $H_J^i(R[It])$ and $H_J^i(gr_I(R))$ are *finitely graded,* that is, are zero in all but finitely many graded pieces. In the case J is the homogeneous maximal ideal, this can be done using local duality (see [HM2]). In this paper we use a theorem of Faltings on annihilation of local cohomology to determine the smallest integer i such that $H_J^i(M)$ is not finitely graded, where J is any homogeneous ideal and M any finitely generated graded module of a graded ring G (Corollary 2.8). In the last section we apply Corollary 2.8 to the graded rings $R[It]$ and $gr_I(R)$ to prove Theorem 3.4.

All rings in this paper are assumed to be commutative with identity. With few exceptions, we adopt the same notation and conventions as in [HM2]. For basic definitions and results on local cohomology, we refer the reader to [Gr] or [GW]. We will make frequent use of the fact that if R is a Noetherian ring, I is an ideal of R, and M is a finitely generated R-module, then $\operatorname{depth}_I M = \min\{i \in \mathbf{Z} \mid H_I^i(M) \neq 0\}$. In particular, $\operatorname{depth}_I M = \infty$ if $I = R$ or $M = 0$.

2. Finitely graded local cohomology

Throughout this section, G will denote a nonnegatively graded Noetherian ring. We call a graded G-module M *finitely graded* if $M_n = 0$ for all but finitely many n, where M_n denotes the nth graded piece of M. We make a couple elementary observations:

Remark 2.1. If M is a finitely graded G-module, then $G_+ \subset \sqrt{\operatorname{Ann}_G M}$. If M is finitely generated, then the converse is true.

Remark 2.2. If M is a finitely graded G-module, then $H_I^i(M)$ is finitely graded for all i and all homogeneous ideals I of G.

Proof. By (2.1), $G_+ \subset \sqrt{\operatorname{Ann}_G M}$, so by passing to $G/\operatorname{Ann}_G M$ we may assume G is finitely graded. Hence, the localization of M at any multiplicatively closed set of homogeneous elements is finitely graded. Since $H_I^i(M)$ can be computed by taking homology of an appropriate Čech complex, we see that $H_I^i(M)$ is finitely graded.

The converse to Remark 2.2 is not true even if M is finitely generated. However, the converse does hold in the case G_0 is local and M is finitely generated. See Corollary 2.6 and the ensuing remarks.

For a given graded G-module M and homogeneous ideal I of G, an interesting (yet difficult) problem is to determine the integers i for which $H_I^i(M)$ is finitely graded. In the next proposition, we give a partial solution to this problem by pinpointing the smallest i such that $H_I^i(M)$ is *not* finitely graded. In preparation for the proof of this result, we establish the following notation:

$$g_I(M) := \sup\{k \in \mathbf{Z}_{\geq 0} \mid H_I^i(M) \text{ is finitely graded for all } i < k\}$$

and

$$t_I(M) := \sup\{k \in \mathbf{Z}_{\geq 0} \mid G_+ \subset \sqrt{\operatorname{Ann}_G H_I^i(M)} \text{ for all } i < k\}.$$

Here, $\mathbf{Z}_{\geq 0}$ denotes the set of nonnegative integers.

Proposition 2.3 (cf. [TI], Lemma 2.2). *Let M be a finitely generated graded G-module and I a homogeneous ideal. Then $g_I(M) = t_I(M)$.*

Proof. By Remark 2.1, $g_I(M) \leq t_I(M)$. To show $t_I(M) \leq g_I(M)$ we make several reductions. First, we may assume that $G_+ \not\subset \sqrt{\operatorname{Ann}_G M}$. For if $G_+ \subset \sqrt{\operatorname{Ann}_G M}$, then by Remarks 2.1 and 2.2, $g_I(M) = \infty$. Let $P_1, \dots, P_r$ be the associated primes of M which do not contain G_+, and let x be a homogeneous element of G_+ not contained in any P_i for $i = 1, \dots, r$. Then x is superficial for M ([ZS]); that is, $(0 :_M x)$ is finitely graded. Now consider the exact sequence

$$0 \longrightarrow (0 :_M x) \longrightarrow M \longrightarrow M/(0 :_M x) \longrightarrow 0.$$

By Remark 2.2, $g_I((0 :_M x)) = \infty$, and consequently $t_I((0 :_M x)) = \infty$. From the long exact sequence on local cohomology, we see that

$$g_I(M) = g_I(M/(0 :_M x)) \quad \text{and} \quad t_I(M) = t_I(M/(0 :_M x)).$$

Thus, by replacing M by $M/(0 :_M x)$, we may assume that x is not a zero-divisor on M. Also, since $x \in G_+$, there exists $e \geq 1$ such that x^e annihilates $H_I^i(M)$ for all $i < t_I(M)$. So by replacing x with x^e, we can assume that $xH_I^i(M) = 0$ for all $i < t_I(M)$.

We now show that $t_I(M) \leq g_I(M)$. We'll use induction to prove that if $0 \leq k \leq t_I(M)$, then $k \leq g_I(M)$. If $k = 0$ there's nothing to show, so suppose $0 < k \leq t_I(M)$ and $k - 1 \leq g_I(M)$. Let $\ell = \deg x$ and consider the short exact sequence

$$0 \longrightarrow M(-\ell) \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0.$$

From the long exact sequence on local cohomology, we have that $k - 1 \leq t_I(M) - 1 \leq t_I(M/xM)$. By induction, $k - 1 \leq g_I(M/xM)$. In particular, $H_I^{k-2}(M/xM)$ is finitely graded. Therefore

$$0 \longrightarrow H_I^{k-1}(M)_n \xrightarrow{x} H_I^{k-1}(M)_{n+\ell}$$

is exact for all but finitely many n. But $xH_I^{k-1}(M) = 0$. Hence $H_I^{k-1}(M)$ is finitely graded and $k \leq g_I(M)$.

We now recall a theorem of Faltings concerning the annihilation of local cohomology. Let S be a Noetherian ring, M a finitely generated S-module and I, J two ideals of S. Define

$$s(I, J, M) := \sup\{k \in \mathbf{Z}_{\geq 0} \mid J \subset \sqrt{\operatorname{Ann}_S H_I^i(M)} \text{ for all } i < k\}.$$

Faltings ([F], Satz 1) proves that if S is the homomorphic image of a regular ring, then

$$s(I, J, M) = \min_{\substack{p, q \in \operatorname{Spec}(S) \\ p \not\supset J \\ q \supset I + p}} \{\operatorname{depth} M_p + \operatorname{ht}(q/p)\}.$$

If the set over which this minimum is taken is empty, we set $s(I, J, M) = \infty$. The following proposition shows that if S, M, and I are all graded, then only the homogeneous primes of S are relevant in computing $s(I, J, M)$. Let $\operatorname{HSpec}(G)$ denote the set of all homogeneous primes of G.

Proposition 2.4. *Let G be a Noetherian graded ring which is the homomorphic image of a regular ring. Let I be a homogeneous ideal of G, J any ideal of G and M a finitely generated graded G-module. Then*

$$s(I, J, M) = \min_{\substack{p, q \in \mathrm{HSpec}(G) \\ p \not\supset J \\ q \supset I+p}} \{\operatorname{depth} M_p + \operatorname{ht}(q/p)\}.$$

Proof. Let $s(I, J, M) = \operatorname{depth} M_p + \operatorname{ht}(q/p)$ where $p \not\supset J$ and $q \supset I + p$. Suppose p is not homogeneous. Let p^* be the ideal of G generated by all the homogeneous elements of p. Then $\operatorname{depth} M_p = \operatorname{depth} M_{p^*} + 1$ and $\operatorname{ht}(q/p) = \operatorname{ht}(q/p^*) - 1$ (see [GW], for example). Thus, $s(I, J, M) = \operatorname{depth} M_{p^*} + \operatorname{ht}(q/p^*)$. Of course, $p^* \not\supset J$ and $q \supset I + p^*$. Since q must be minimal over $I + p^*$, q is homogeneous.

Combining Propositions 2.3 and 2.4, we obtain the following corollary.

Corollary 2.5. *Let G be a nonnegatively graded Noetherian ring which is the homomorphic image of a regular ring. Let I be a homogeneous ideal of G and M a finitely generated graded G-module. Then*

$$g_I(M) = \min_{\substack{p, q \in \mathrm{HSpec}(G) \\ p \not\supset G_+ \\ q \supset I+p}} \{\operatorname{depth} M_p + \operatorname{ht}(q/p)\}.$$

We now prove a partial converse to Remark 2.2.

Corollary 2.6. *Let G be a nonnegatively graded Noetherian ring such that G_0 is local. Let $I \neq G$ be a homogeneous ideal of G and M a finitely generated graded G-module. Then $g_I(M) = \infty$ if and only if M is finitely graded.*

Proof. If M is finitely graded, then $g_I(M) = \infty$ by Remark 2.2. To prove the converse, we first note that by passing to the ring $\hat{G} = G \otimes_{G_0} \hat{G}_0$ (where $\hat{G}_0$ is the completion of G_0 with respect its maximal ideal), we may assume that G_0 (and hence G) is the homomorphic image of a regular ring. Now suppose $p \in \mathrm{HSpec}(G)$ and $p \supset \operatorname{Ann}_G M$. Then $\operatorname{depth} M_p < \infty$. Using Corollary 2.5 and the assumption that $g_I(M) = \infty$, we see that $p \supset G_+$ or $p + I = G$. But $p + I$ is contained in the unique homogeneous maximal ideal of G, so we must have that $p \supset G_+$. Thus, $G_+ \subset \sqrt{\operatorname{Ann}_G M}$ and M is finitely graded by Remark 2.1.

We remark that the above corollary does not hold if G_0 is not local. For example, let R be any regular ring which is not connected and $J = (e)$ where e is a nontrivial idempotent of R. Let $G = R[Jt]$ and $I = JG$. Then one sees that $g_I(G) = \infty$ (by using Corollary 2.5, for instance), but G is not finitely graded.

3. Applications to the Rees algebra and the associated graded ring

Let R be a Noetherian ring and I an ideal of R. In this section, we apply the ideas developed in the previous section to establish some equalities and inequalities between depths of R, $gr_I(R)$ and $R[It]$ with respect to any homogeneous ideal of $R[It]$ containing $IR[It]$ (Theorem 3.4). Here, we will view R and $gr_I(R)$ as graded $R[It]$-modules, where $R = R[It]/ItR[It]$ and $gr_I(R) = R[It]/IR[It]$. Note that for any ideal J of $R[It]$ we have that $\operatorname{depth}_J R = \operatorname{depth}_{J\cap R} R$ and $\operatorname{depth}_J gr_I(R) = \operatorname{depth}_{\overline{J}} gr_I(R)$, where $\overline{J} = (J + IR[It])/IR[It]$.

The following two lemmas will be needed in the proof of Proposition 3.3. Proofs of these facts can be found in [HM2].

Lemma 3.1. *Let $S = R[It]$ and $G = gr_I(R)$. Suppose $p \in \operatorname{Spec}(S)$, $p \supset IS$, $p \not\supset S_+$. Then $\operatorname{depth} S_p = \operatorname{depth} G_p + 1$.*

Lemma 3.2. *Suppose R is catenary, and let $p_1 \subset p_2 \subset q$ be primes of R. Then for any finitely generated R-module M*

$$\operatorname{depth} M_{p_1} + \operatorname{ht}(q/p_1) \geq \operatorname{depth} M_{p_2} + \operatorname{ht}(q/p_2).$$

The next proposition is a generalization of Proposition 3.2 of [HM2]. It serves as the cornerstone for the proof of Theorem 3.4.

Proposition 3.3. *Suppose that R is the homomorphic image of a regular ring and I is an ideal of R. Let $S = R[It]$ and $G = gr_I(R)$. Let J be a homogeneous ideal of S which contains IS. Then*

$$g_J(S) = g_J(G) + 1.$$

Proof. Since R is the homomorphic image of a regular ring, so is S and we may use Corollary 2.5. We first show that $g_J(S) \leq g_J(G) + 1$. This is trivial if $g_J(G) = \infty$, so we assume that $g_J(G) < \infty$. Then $g_J(G) = \operatorname{depth} G_p + \operatorname{ht}(q/p)$ for some $p, q \in \operatorname{HSpec}(S)$ with $p \not\supset S_+$ and $q \supset J + p$. As $G_p \neq 0$, $p \supset IS$. By Lemma 3.1,

$$\begin{aligned} g_J(S) &\leq \operatorname{depth} S_p + \operatorname{ht}(q/p) \\ &= \operatorname{depth} G_p + 1 + \operatorname{ht}(q/p) \\ &= g_J(G) + 1. \end{aligned}$$

It now suffices to show that $g_J(S) \geq g_J(G) + 1$. Again, this holds trivially if $g_J(S) = \infty$, so we assume that $g_J(S) < \infty$. Thus, $g_J(S) = \operatorname{depth} S_p + \operatorname{ht}(q/p)$ for some $p, q \in \operatorname{HSpec}(S)$ such that $p \not\supset S_+$ and $q \supset J + p$.

Claim. *There exists $p' \in \operatorname{HSpec}(S)$ such that $q \supset p' \supset (IS, p)$ and $p' \not\supset S_+$.*

Proof. Let $q_0 = q \cap R$. By passing to the ring $S \otimes_R R_{q_0} \cong R_{q_0}[I_{q_0}t]$, it suffices to prove the claim in the case q_0 is the unique maximal ideal of R. Since the statement is trivial if $q \not\supset S_+$ (take $p' = q$), we may assume q is the unique homogeneous maximal ideal of S. Suppose the claim is false. Then $(S_+)_q \subset \sqrt{(IS, p)_q}$. Since p and IS are homogeneous, this implies $S_+ \subset \sqrt{(IS, p)}$. Therefore, $I^n = I^{n+1} + p_n$ for n sufficiently large (where $p = \bigoplus p_n t^n$). By Nakayama's lemma, $I^n = p_n$ for sufficiently large n and hence $p \supset S_+$, a contradiction. This proves the claim.

By Lemma 3.2, $g_J(S) \leq \operatorname{depth} S_{p'} + \operatorname{ht}(q/p') \leq \operatorname{depth} S_p + \operatorname{ht}(q/p) = g_J(S)$. Thus, by replacing p with p', we may assume $p \supset IS$. By Lemma 3.1,

$$\begin{aligned} g_J(S) &= \operatorname{depth} S_p + \operatorname{ht}(q/p) \\ &= \operatorname{depth} G_p + 1 + \operatorname{ht}(q/p) \\ &\geq g_J(G) + 1. \end{aligned}$$

Before proving our main result, we make the trivial observation that $g_J(M) \geq \operatorname{depth}_J M$ for any finitely generated graded $R[It]$-module M and homogeneous ideal J of $R[It]$.

Theorem 3.4. *Let (R, m) be a local ring, I an ideal of R and J a homogeneous ideal of $R[It]$ which contains $IR[It]$. Then*

(a) $\operatorname{depth}_J gr_I(R) \leq \operatorname{depth}_J R[It]$,
(b) $\operatorname{depth}_J gr_I(R) \leq \operatorname{depth}_{J\cap R} R$, *and*
(c) *if* $\operatorname{depth}_J gr_I(R) < \operatorname{depth}_{J\cap R} R$, *then* $\operatorname{depth}_J R[It] = \operatorname{depth}_J gr_I(R)+1$.

Proof. Let $S = R[It]$, $G = gr_I(R)$, $s = \operatorname{depth}_J R[It]$, $g = \operatorname{depth}_J gr_I(R)$ and $r = \operatorname{depth}_{J\cap R} R$. By passing to the m-adic completion of R, we may assume R is the homomorphic image of a regular ring. Consider the following two exact sequences:

$$0 \longrightarrow S_+ \longrightarrow S \longrightarrow R \longrightarrow 0, \tag{3.5}$$

$$0 \longrightarrow IS \longrightarrow S \longrightarrow G \longrightarrow 0. \tag{3.6}$$

Using the bottom sequence, we obtain that

$$H_J^i(IS)_n \cong H_J^i(S)_n \qquad \text{for } i \leq g-1 \text{ and all } n. \tag{3.7}$$

Using the top sequence and the fact that $H_J^i(R)_n = 0$ for all $i < r$ or $n \neq 0$, we obtain that

$$H_J^i(S_+)_n \cong H_J^i(S)_n \qquad \text{for all } i < r \text{ or } n \neq 0. \tag{3.8}$$

Since $IS \cong S_+(1)$, we see that

$$H_J^i(IS)_n \cong H_J^i(S_+)_{n+1} \qquad \text{for all } i,\ n. \tag{3.9}$$

Therefore,

$$H_J^i(S)_n \cong H_J^i(S)_{n+1} \qquad \text{for all } i \leq g-1 \text{ and } n \neq -1.$$

But $g_J(S) = g_J(G) + 1 \geq g+1$ which implies that $H_J^i(S)$ is finitely graded for all $i \leq g$. Hence, $H_J^i(S) = 0$ for all $i \leq g-1$, and so $g \leq s$.

To prove (b), suppose that $r < g$. Since $g \leq s$, we know that $H_J^r(S) = 0$. Applying this to the long exact sequence on local cohomology arising from (3.5), we get that $H_J^{r+1}(S_+)_0 \neq 0$. By (3.9), this means that $H_J^{r+1}(IS)_{-1} \neq 0$. Using (3.6) and the fact that $r < g$, we see that

$$0 \longrightarrow H_J^{r+1}(IS)_n \longrightarrow H_J^{r+1}(S)_n \tag{3.10}$$

is exact for all n. In particular, $H_J^{r+1}(S)_{-1} \neq 0$. Combining (3.8), (3.9) and (3.10), there exists an injective map $H_J^{r+1}(S)_n \to H_J^{r+1}(S)_{n-1}$ for all $n \leq -1$. Thus, $H_J^{r+1}(S)$ is not finitely graded. Therefore,

$$g_J(S) \leq r+1 < g+1 \leq g_J(G)+1,$$

contradicting Proposition 3.3. Hence, $g \leq r$.

Finally, we suppose that $g < r$. Then from (3.6) we know there exists an injective map $H_J^i(IS)_n \to H_J^i(S)_n$ for all $i \leq g$ and all n. Using (3.5) and our assumption that $g < r$, we see that $H_J^i(S_+)_n \cong H_J^i(S)_n$ for all $i \leq g$ and all n. Thus, there exist injective maps

$$H_J^i(S)_{n+1} \to H_J^i(S)_n$$

for all $i \leq g$ and all n. Since $g_J(S) = g_J(G) + 1 \geq g+1$, $H_J^i(S) = 0$ for all $i \leq g$. Thus $s \geq g+1$. But from the exact sequences (3.5) and (3.6), we get that $H^i_J(G) \cong H^i_{J\cap R}(R)$ for $i \leq s-2$. So if $s \geq g+2$, then $g=r$, contradicting that $g<r$. Therefore, $s=g+1$.

Remark 3.11. The proof shows that Theorem 3.4 holds when R is any Noetherian ring (not necessarily local) which is the homomorphic image of a regular ring.

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