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THE TEICHMULLER FLOW IS HAMILTONIAN

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ABSTRACT. It is shown that the Teichmuller flow on the cotangent bundle over Teichmuller space coincides with the Hamiltonian flow defined by the function which gives the length of a cotangent vector.

INTRODUCTION

Suppose M is a smooth manifold with local coordinates $(q_1, \dots, q_n)$. Then the set of 1 forms $dq_1, \dots, dq_n$ form a basis for the cotangent space at each point and so any cotangent vector v^* can be written as $p_1dq_1 + \dots + p_ndq_n$ for coefficients $p_1, \dots, p_n$. Then $(q_1, \dots, q_n, p_1, \dots, p_n)$ are symplectic coordinates for the cotangent bundle CTM. Any smooth function $H: CTM \to R$ defines Hamilton's equations:

$$dq_i/dt = \partial H/\partial p_i$$

and

$$dp_i/dt = -\partial H_i/\partial q_i.$$

The corresponding flow is called the Hamiltonian flow. Suppose M has a Riemannian metric and $H(v^*) = |v^*|^2/2$ where $|v^*|$ is its length. It is a classical result [2, p. 53] that the Hamiltonian flow and the geodesic flow on CTM coincide.

In this paper we consider the Teichmuller space T_g of closed Riemann surfaces of genus $g \geq 2$. It is a fundamental result that T_g is a complex manifold and that the cotangent space at a point $X \in T_g$ is the vector space $Q(X)$ of holomorphic quadratic differentials on X. The Teichmuller space also comes equipped with the Teichmuller metric which is not Riemannian, but rather a Finsler metric, which means it is defined by a norm on the tangent space and a dual norm

$$||\phi|| = \int_X |\phi(z)\, dz^2|$$

on the cotangent space $Q(X)$. Thus the standard equations of Riemannian geometry are not available. Nonetheless the geodesics in this metric are well understood. The geodesics are determined by the family of Teichmuller extremal

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maps defined by a fixed quadratic differential and a 1 parameter family of real numbers. At the level of the cotangent bundle $\mathscr{Q}$ this leads to a flow called the Teichmuller flow. The question then arises whether this flow is Hamiltonian for the corresponding length function

$$H(\phi) = \frac{||\phi||^2}{2}$$

as in the classical case.

An immediate difficulty arises from consideration of quadratic differentials with higher order zeroes. A result of Royden's [7] says that the vector field

$$(\partial H/\partial p_i, -\partial H/\partial q_i)$$

is not Lipschitz at quadratic differentials with zeroes of order at least 3. Thus the Hamiltonian system may not admit a unique solution. Because of this difficulty we define $\mathscr{Q}_1 \subset \mathscr{Q}$ to be the subset of quadratic differentials with only simple zeroes. This is a dense subset of $\mathscr{Q}$ and is known as the principal stratum. The Teichmuller flow preserves $\mathscr{Q}_1$. Our Theorem states

Theorem. *The Hamiltonian flow is C^∞ on $\mathscr{Q}_1$ and coincides with the Teichmuller flow. On $\mathscr{Q} - \mathscr{Q}_1$, the Teichmuller flow satisfies Hamilton's equations.*

In the next section we will show that the flows are C^∞ on $\mathscr{Q}_1$. We will then introduce coordinates for T_g that allow us to show that Hamilton's equations are satisfied along the Teichmuller flow lines in $\mathscr{Q}_1$. Continuity will allow us to conclude that the Teichmuller flow on $\mathscr{Q} - \mathscr{Q}_1$ also satisfies Hamilton's equations. In particular this means that at a point on a lower dimensional stratum, the Hamiltonian vector field is tangent to that stratum. However we do not know if the Hamiltonian system is C^1 along that stratum and therefore do not know if there are other solutions to the Hamiltonian system other than the Teichmuller flow.

Coordinates for Teichmuller space

A Riemann surface X can be described by a family $\{U_\mu, z_\mu\}$, where the U_μ form an open cover of X and $z_\mu : U_\mu \to C$ are homeomorphisms such that $z_\mu \circ z_\nu^{-1}$ is analytic whenever defined. The maps z_μ are called local uniformizers. A holomorphic quadratic differential $\phi(z)dz^2$ on X assigns to each local uniformizer z_μ a holomorphic function $\phi_\mu(z_\mu)$ such that in the overlap

$$\phi_\mu(z_\mu)dz_\mu^2 = \phi_\nu(z_\nu)dz_\nu^2.$$

Associated to a quadratic differential are the horizontal and vertical trajectories. These are the arcs along which $\phi(z)dz^2 > 0$ and $\phi(z)dz^2 < 0$ respectively. The set of horizontal and vertical trajectories forms the horizontal and vertical foliations. We denote the latter by $v(\phi)$. A quadratic differential ϕ also defines a metric $|\phi^{1/2}(z)dz|$ which is locally Euclidean except at the zeroes of ϕ which are singularities of the metric. The set of all quadratic differentials on X forms a complex vector space $Q(X)$ of dimension $3g-3$. As X varies over the Teichmuller space T_g, these vector spaces fit together to form a bundle $\mathscr{Q}$ over T_g. A Beltrami differential on X assigns to each uniformizer z a measurable function $\mu(z)$ such that

$$\mu(z)\frac{d\bar{z}}{dz}$$

is invariant under changes of coordinates. Then $|\mu(z)|$ defines a function on X. There is a pairing between $Q(X)$ and the space $M(X)$ of L^∞ Beltrami differentials on X given by

$$\langle \phi, \mu \rangle = \mathrm{Re} \int_X \phi\mu.$$

The infinitessimally trivial Beltrami differentials $M_0(X)$ are those μ for which $\langle \phi, \mu \rangle = 0$ for all $\phi \in Q(X)$. It is a classical result in Teichmuller theory that T_g is a complex manifold, the tangent space at X is $M(X)/M_0(X)$ and $Q(X)$ is the cotangent space at X. We let $\pi : \mathscr{Q} \to T_g$ be the natural projection.

Each $\phi \in \mathscr{Q}$ determines certain topological data $\kappa = (k_1, \dots, k_n; \epsilon = \pm 1)$ where $k_1, \dots, k_n$ are the orders of the zeroes: $\epsilon = +1$ if ϕ is the square of an abelian differential; $\epsilon = -1$ if it is not. A stratum $\mathscr{Q}_\kappa$ consists of all quadratic differentials determining the data κ. The principle stratum $\mathscr{Q}_1$ corresponds to $\kappa = (1, \dots, 1; -1)$ and its complement has codimension 1.

The quantity

$$\int_X |\phi(z) dz^2|$$

which defines the Teichmuller cometric is also the area of the quadratic differential. The geodesics in the Teichmuller metric are defined by the Teichmuller maps. For each $\phi \in Q(X)$ and $t \in R$ the Teichmuller map $f_{\phi, t}$ maps X to a new Riemann surface X_t. There is a quadratic differential ϕ_t on X_t with the property that f_t sends horizontal trajectories of ϕ to horizontal trajectories of ϕ_t expanding lengths by a factor of e^t and sends vertical trajectories to vertical trajectories contracting lengths by the same factor e^t. We can take ϕ_t so that $H(\phi_t) = H(\phi)$. It also sends zeroes of ϕ to zeroes of ϕ_t of the same order. At the level of the cotangent bundle $\mathscr{Q}$ this gives a flow $\phi \to \phi_t$ called the Teichmuller flow and the flow preserves each stratum $\mathscr{Q}_\kappa$. These flows have been studied in [5], [8], and [6].

For any ϕ_0 in the principle stratum $\mathscr{Q}_1$ we may triangulate the underlying surface so that the edges of the triangulations are geodesic segments with respect to the metric $|\phi_0(z)^{1/2} dz|$ and the vertices are zeroes of ϕ_0. (A canonical triangulation is given in [6].) Each ϕ near ϕ_0 in $\mathscr{Q}_1$ has a corresponding triangulation by geodesic edges. For ϕ in a neighborhood of ϕ_0 we may continuously choose a branch of $\phi^{1/2}$ along each edge. To each directed edge e is associated a holonomy vector $\mathrm{hol}(e)$ whose components

$$\mathrm{hol}_1(e) = \int_e \mathrm{Re}(\phi^{1/2} dz)$$

and

$$\mathrm{hol}_2(e) = \int_e \mathrm{Im}(\phi^{1/2} dz)$$

are called the horizontal and vertical components of e. The holonomy vectors of a set of $6g-6$ edges serve as analytic coordinates for $\mathscr{Q}_1$ near ϕ_0. The area of a triangle in R^2 is an analytic function of the coordinates of its vertices. Therefore H is an analytic function on $\mathscr{Q}_1$ and Hamilton's equations must have a *unique* solution in a neighborhood of a point in $\mathscr{Q}_1$.

In the holonomy coordinates the Teichmuller flow $(\phi, t) \to \phi_t$ is given by

$$\begin{aligned}&(\mathrm{hol}_1(e_1), \mathrm{hol}_2(e_1), \dots, \mathrm{hol}_1(e_{6g-6}), \mathrm{hol}_2(e_{6g-6}), t)\\ &\to (e^t\mathrm{hol}_1(e_1), e^{-t}\mathrm{hol}_2(e_1), \dots, e^t\mathrm{hol}(e_{6g-6}), e^{-t}\mathrm{hol}_2(e_{6g-6})),\end{aligned} \tag{1.1}$$

and thus is analytic.

Now fix $\phi_0 \in \mathscr{Q}_1$ which determines the flow line $\phi_t \in \mathscr{Q}_1$. Let X_t be the corresponding Teichmuller geodesic through X_0. The proof that Hamilton's equations are satsified along ϕ_t depends on finding a useful set of coordinates in a neighborhood U of X_t. Recall that $v(\phi)$ denotes the vertical measured foliation of the quadratic differential ϕ.

Proposition 1. *There are C^∞ coordinates $(q_1, \dots, q_{6g-6})$ in a neighborhood U of X_t such that*

(1) *for fixed q_1, each point with coordinates $(q_1, \dots, q_{6g-6})$ has a quadratic differential ϕ such that $H(\phi) = 1$ and $v(\phi) = e^{q_1}v(\phi_0)$;*

(2) *for fixed $(q_2, \dots, q_{6g-6})$, the points with coordinates $(q_1, q_2, \dots, q_{6g-6})$ parametrize a Teichmuller geodesic with q_1 as arclength parameter.*

Proof. Let $F = v(\phi_0)$ be the vertical foliation of ϕ_0. We let

$$E_F = \{\phi \in \mathscr{Q} : v(\phi) = F\}.$$

Then $E_F \cap \mathscr{Q}_1$ is locally described near ϕ_0 by a set of equations

$$\mathrm{hol}_1(e_i) = \text{constant}$$

and thus is a smooth submanifold of $\mathscr{Q}_1$. In particular H restricted to $E_F \cap \mathscr{Q}_1$ is smooth. Moreover by the Main Theorem of [4] the projection

$$\pi : E_F \to T_g$$

is a local diffeomorphism at ϕ_0. (The Main Theorem of [4] says that the map is a homeomorphism. The proof uses the inverse function theorem. The fact that the derivative at ϕ_0 is an isomorphism is proved in Lemma 4.4 and Proposition 4.16.) Then $E_F \cap \mathscr{Q}_1 \cap H^{-1}(1)$ is a smooth submanifold of $\mathscr{Q}_1$ near ϕ_0 which maps diffeomorphically onto its image N_{ϕ_0} which is a codimension 1 submanifold of T_g. Find local coordinates $(q_2, \dots, q_{6g-6})$ for N_{ϕ_0}, with 0 corresponding to X_0. We now define a map f from a neighborhood of 0 in R^{6g-6} into T_g. Given $(q_1, q_2, \dots, q_{6g-6})$ let $X \in N_{\phi_0}$ have coordinates $(q_2, \dots, q_{6g-6})$ and let $\phi \in E_F \cap \mathscr{Q}_1 \cap H^{-1}(1)$ be such that $\pi(\phi) = X$. Then let ϕ_{q_1} be the quadratic differential found by flowing time q_1 from ϕ. Set

$$f(q_1, \dots, q_{6g-6}) = \pi(\phi_{q_1}).$$

If we can show that f is a local diffeomorphism, then $(q_1, \dots, q_{6g-6})$ will serve as local coordinates for T_g near X_t. Since $v(\phi_{q_1}) = e^{q_1}v(\phi_0)$, these coordinates will satisfy (1) and (2). To see that f is smooth note that f can be written as a composite

$$(q_1, \dots, q_{6g-6}) \to (q_1, \phi) \to \phi_{q_1} \to \pi(\phi_{q_1})$$

of smooth maps. We now show that Df is an isomorphism at 0 and then apply the inverse function theorem.

First we note that for $i \geq 2$, $\mu_i = Df(0)(\partial/\partial q_i)$ are independent vectors in the tangent space to N_{ϕ_0} at X_0. Thus we need to prove that $\mu_1 = Df(0)(\partial/\partial q_1)$

is a nonzero vector that is not tangent to N_{ϕ_0}. But μ_1 is a unit vector tangent to the Teichmuller geodesic determined by ϕ_0. Thus $\mu_1 = \frac{\bar{\phi}_0}{|\phi_0|}$ and so

$$\langle \phi_0, \mu_1 \rangle = 1.$$

We now rely on a result from [3]. We introduce a function $G : T_g \to R$. For each $X \in T_g$ by the Main Theorem of [4] there exists a unique $\psi \in Q(X)$ such that $v(\psi) = F$. Define

$$G(X) = \log ||\psi||.$$

Then

$$G^{-1}(G(X_0)) = G^{-1}(\log||\phi_0||) = G^{-1}(0) = N_{\phi_0}.$$

Then ([3], Theorem 5, p. 217) G is smooth and the derivative of G at X in the direction of μ is given by the formula

$$DG(X)[\mu] = 2\langle \psi, \mu \rangle = \operatorname{Re} \int_X 2\mu\psi.$$

Since $G = 0$ on N_{ϕ_0},

$$\langle \phi_0, \mu \rangle = 0$$

for all μ tangent to N_{ϕ_0}. Since $\langle \phi_0, \mu_1 \rangle = 1$, μ_1 is not tangent to N_{ϕ_0}. $\square$

Proof of Theorem

We begin by proving that Hamilton's equations are satisfied along each Teichmuller geodesic in the principle stratum $\mathcal{Q}_1$. Introduce the coordinates $(q_1, \dots, q_{6g-6})$ in a neighborhood of X_0 given by Proposition 1. They define symplectic coordinates

$$(q_1, \dots, q_{6g-6}, p_1, \dots, p_{6g-6})$$

for $\mathcal{Q}$ in a neighborhood of ϕ_0. First let $\phi \in E_F \cap \mathcal{Q}_1 \cap H^{-1}(1)$. Then $\pi(\phi)$ has coordinates $(0, q_2, \dots, q_{6g-6})$. An argument similar to that given in Proposition 1 shows that the coordinates of ϕ_{q_1} are

$$(q_1, \dots, q_{6g-6}, 1, \dots, 0). \tag{2.1}$$

For by construction, the q coordinates are $q_1, \dots, q_{6g-6}$. For each t let $F_t = v(\phi_t) = e^t v(\phi)$. For each $X \in T_g$ let

$$G_t(X) = \log ||\psi||,$$

where $\psi \in Q(X)$ is the unique quadratic differential such that $v(\psi) = F_t$. Then $G_t = 0$ on the fiber $\{(q_1, \dots, q_{6g-6}) : q_1 = t\}$, so

$$\langle \phi_t, \mu \rangle = 0$$

for all μ tangent to the fiber or, in other words for $i \geq 2$,

$$\langle \phi_t, \partial/\partial q_i \rangle = 0.$$

This implies $\phi_t = r\,dq_1$ for $r \in R$. Since $\partial/\partial q_1$ is tangent to the Teichmuller geodesic in the direction of positive time, in fact $r > 0$. However since $\mu_1 = \partial/\partial q_1$ is a unit vector in the Teichmuller metric, $H(dq_1) = 1$. Since $H(\phi_t) =$

1, $\phi_t = dq_1$ and so ϕ_t has coordinates $(t, q_2, \ldots, q_{6g-6}, 1, 0 \ldots, 0)$, proving (2.1). We also note that

$$\mu_1 = \frac{\bar{\phi}_t}{|\phi_t|}.$$

Now let $\phi_{0,t}$ be the path through ϕ_0, so by (2.1) it has coordinates

$$(t, 0, \ldots, 0, 1, 0, \ldots, 0).$$

Then along $\phi_{0,t}$,

$$dq_1/dt = 1, \; dp_1/dt = 0, \; dq_i/dt = dp_i/dt = 0, \; i \neq 1. \tag{2.2}$$

Since $H(\phi_{q_1}) = 1$ where ϕ_{q_1} has coordinates $(q_1, q_2, \ldots, q_{6g-6}, 1, \ldots, 0)$,

$$\partial H/\partial q_i(t, 0, \ldots, 0, 1, 0, \ldots, 0) = 0. \tag{2.3}$$

Since $H((1+s)\phi_{0,t}) = \frac{(1+s)^2}{2} H(\phi_{0,t}) = \frac{(1+s)^2}{2}$, we have

$$\partial H/\partial p_1(t, 0, \ldots, 0, 1, 0, \ldots, 0) = \frac{1}{2}\frac{d(1+s)^2}{ds}(0) = 1. \tag{2.4}$$

Finally we apply a formula of Royden's [7]. For $\chi, \psi \in Q(X)$,

$$\frac{d\|\chi + t\psi\|}{dt}(0) = \operatorname{Re}\int_X \psi \frac{\bar{\chi}}{|\chi|}.$$

This is applied with

$$\chi = \phi_{0,t} = (t, 0, \ldots, 0, 1, 0, \ldots, 0)$$

and

$$\psi = dq_i = (t, 0, \ldots, 0, 0, \ldots, 1, \ldots, 0).$$

Since $\|\phi_{0,t}\| = 1$ and $\frac{\bar{\phi}_{0,t}}{|\phi_{0,t}|} = \mu_1 = \partial/\partial q_1$, for $i \geq 2$,

$$\begin{aligned} \partial H/\partial p_i(t, 0, \ldots, 0, 1, 0, \ldots, 0) &= \frac{d}{ds}\|\phi_{0,t} + s\,dq_i\|(s=0) \\ &= \operatorname{Re}\int dq_i \mu_1 = \langle dq_i, \partial/\partial q_1 \rangle = 0. \end{aligned} \tag{2.5}$$

We conclude from (2.2) and (2.3)–(2.5) that Hamilton's equations are satisfied along $\phi_{0,t}$.

To finish the proof of the Theorem we need to discuss the lower dimensional strata $\mathscr{Q}_\kappa$. We begin by recalling some results proved in [4]. Suppose $q_0 \in \mathscr{Q}_\kappa$ is a quadratic differential on the Riemann surface X. Let Λ_{q_0} be the sheaf of germs of vector fields χ such that

$$q_0(\chi, \chi) = \text{constant}.$$

For $k \geq 2$ let P_k be the set of polynomials of the form

$$z^k + a_{k-2}z^{k-2} + \ldots + a_0,$$

and S_k the set of polynomials of the form

$$a_{k-2}z^{k-2} + \ldots + a_0,$$

the tangent space to P_k at z^k. Suppose ϕ_0 has zeroes of order $k_1, \ldots, k_n$. In a neighborhood of the zero of order k_i there are coordinates z such that

$\phi_0 = z^{k_i} dz^2$. Let U a be small neighborhood of ϕ_0 in $\mathcal{Q}$. There is an analytic map

$$f : U \to \prod_i^n P_{k_i}$$

classifying the deformations of the zeroes of ϕ_0. Then $\mathcal{Q}_\kappa$ is defined near ϕ_0 by

$$f^{-1}(z^{k_1}, \dots , z^{k_n}).$$

If ϕ_0 is not the square of an abelian differential, then by [4], Proposition 4.7, the derivative of f is onto $\bigoplus S_{k_i}$ and there is an exact sequence

$$0 \to H^1(X, \Lambda_{q_0}) \to T_{q_0}\mathcal{Q} \to \bigoplus S_{k_i} \to 0.$$

If ϕ_0 is the square of an abelian differential, choose a small circle γ_i about the zero and define a map $\alpha_i : U \to C$ by $\phi \to \int_{\gamma_i} \phi^{1/2} dz$. Here the branch of $\phi^{1/2}$ is chosen to be near $z^{k_i/2}$ for ϕ near ϕ_0. Then [4], Lemma 4.8, says that the map f is a submersion onto the submanifold defined by the equation $\sum \alpha_i(\phi) = 0$. Now there is an exact sequence

$$0 \to H^1(X, \Lambda_{q_0}) \to T_{q_0}\mathcal{Q} \to \bigoplus S_{k_i} \to C \to 0.$$

In either case the implicit function theorem says that $\mathcal{Q}_\kappa$ is an analytic submanifold of $\mathcal{Q}$; $T_{\phi_0}\mathcal{Q} = T_{\phi_0}\mathcal{Q}_\kappa \oplus S_{k_i}$ in the first case, and $T_{\phi_0}\mathcal{Q} = T_{\phi_0}\mathcal{Q}_\kappa \oplus S$ where S is codimension 1 subspace of $\bigoplus S_{k_i}$ in the second.

Proposition 2. *The Teichmuller flow restricted to $\mathcal{Q}_\kappa$ is real analytic.*

Proof. We may triangulate the underlying surface of ϕ_0 so that the edges are geodesic segments joining the zeroes of ϕ_0 and the triangles have no zeroes in their interior. Let p be the dimension of $\mathcal{Q}_\kappa$. There is a choice of p edges e_i of the triangulation such that the holonomy vectors $\mathrm{hol}_1(e_1), \mathrm{hol}_2(e_1), \dots,$ $\mathrm{hol}_1(e_p), \mathrm{hol}_2(e_p)$ serve as local coordinates for $\mathcal{Q}_\kappa$ near ϕ_0. The Teichmuller flow preserves $\mathcal{Q}_\kappa$ and in terms of the holonomy vectors it is described by (1.1), so is analytic. □

We continue with the proof of the Theorem. Choose ϕ near ϕ_0 which has simple zeroes and such that the critical vertical trajectories of ϕ in each neighborhood of the zeroes of ϕ_0 form a connected set of edges e_i. Again let $f : \mathcal{Q} \to \prod P_{k_i}$ be the map classifying the deformations of the zeroes of ϕ_0. We may express

$$f(\phi(z)) = \prod (z - r_i) dz^2 = (z^k + a_{k-l} z^{k-l} + \dots) dz^2.$$

Let p_s be the family of polynomials

$$p_s = \prod (z - s^{1/l} r_i) = z^k + s a_{k-l} z^{k-l} + \dots ,$$

which converge to z^k as $s \to 0$. It is easy to check by a change of variables that this family also has the property that the critical vertical trajectories also form a connected set of edges e_i. Moreover the holonomy vector $hol_i(s)$ of e_i at time s satisfies

$$\frac{hol_i(s_1)}{hol_i(s_2)} = \left(\frac{s_1}{s_2}\right)^{\left(\frac{k/2+1}{l}\right)},$$

which says in particular that the change in holonomy vector is by a constant factor independent of e_i. From this we see that

$$\lim_{s\to 0}\frac{\frac{d}{ds}hol_i(s)}{hol_i(s)}=\lim_{s\to 0}\frac{(k/2+1)}{ls}=\infty. \tag{2.6}$$

Let $\phi_s\to\phi_0$ a family of quadratic differentials so that

$$f(\phi_s(z))=p_s(z).$$

We may find a set $e_{i,s}$, $i=1,\dots,6g-6$, of edges of ϕ_s whose holonomy vectors serve as local coordinates for $\mathscr{Q}$ near ϕ_s such that for $i\le p$ the edges $e_{i,s}$ converge to the edges e_i of ϕ_0 that determine local coordinates for $\mathscr{Q}_\kappa$ and for $i>p$ are vertical edges in the neighborhood of the zeroes of ϕ_0. Now for each s and t consider the flow $\phi_s\to\phi_{s,t}$. Since Teichmuller maps contract the holonomy of the vertical edges $e_{i,s}$, $i\ge p+1$, in the neighborhood of the zeroes by a constant factor independent of the edge $e_{i,s}$, there must be $s'=s'(s,t)$ such that

$$f(\phi_{s,t})=p_{s'}.$$

Let $v_{s,t}$ be the tangent vector to the flow $\phi_s\to\phi_{s,t}$ at $\phi_{s,t}$. Then $Df(v_{s,t})$ is tangent to the family p_s at $s=s'$. By (1.1) at time s' we have

$$\frac{Dhol_i(s')(Df(v_{s,t}))}{hol_i s'}=-e^{-t} \tag{2.7}$$

and this is independent of s; in particular this quantity does not go to infinity as $s\to 0$. The tangent vector $Df(v_{s,t})$ is a multiple $\lambda(s')$ of the tangent vector to the family p_s at s', and comparing (2.6) and (2.7) we see that $\lambda(s')\to 0$ as $s\to 0$. Thus $Df(v_{s,t})\to 0$ as $s\to 0$ for each t and we conclude that as $s\to 0$ any convergent subsequence of tangent vectors to the flow at $\phi_{s,t}$ converges to a vector tangent to the stratum $\mathscr{Q}_\kappa$; namely an element of $H^1(X,\Lambda_{q_0})$. We may interpret such an element as infinitessimal change in the holonomy of the edges e_i. Since $hol_i(s)\to hol_i(e)$ for $i\le p$, by formula (1.1), the limit must be tangent to the flow through ϕ_0. In other words the tangent vector

$$(dq_1/dt,\dots,dp_{6g-6}/dt)$$

to the flow at $\phi_{s,t}$ converges to the tangent vector to the flow through ϕ_0 at time t as $s\to 0$. The vector field

$$(\partial H/\partial p_i,-\partial H/\partial q_i)$$

is continuous on $\mathscr{Q}$ ([7]). Since Hamilton's equations are satisfied along the flow through ψ_s, by continuity they are satisfied along the flow through ϕ_0. □

From the work of [5], [6], and [8], $\mathscr{Q}_1$ has an absolutely continous measure ρ invariant under the Teichmuller flow and invariant under the action of the mapping class group $\mathrm{Mod}(g)$. In the local coordinates defined by holonomy vectors $\{hol_1(e_j),hol_2(e_j)\}$, $j=1,6g-6$, the measure is described by

$$d\rho=d\ hol_1(e_1)\wedge d\ hol_2(e_1)\wedge\dots\wedge d\ hol_2(e_{6g-6}).$$

Corollary. *We have* $d\rho=dq_1\wedge\dots\wedge dq_n\wedge dp_1\dots\wedge dp_n$.

Proof. The measure $dq_1\wedge\dots\wedge dq_n\wedge dp_1\dots\wedge dp_n$ is absolutely continous with respect to ρ. Each measure is invariant under the Teichmuller flow on $\mathscr{Q}_1/\mathrm{Mod}(g)$. Since ρ is an ergodic measure for the flow [5], [8], the measures must be equal. □

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