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ADDING SMALL SETS TO AN N-SET

ZUZANA BUKOVSKÁ AND LEV BUKOVSKÝ

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ABSTRACT. Pseudo Dirichlet and N-sets are small sets of reals defined in the theory of trigonometric series. We prove that by adding a set of cardinality smaller than $\mathfrak{p}$ to an N-set one obtains again an N-set. This is a strengthening of Arbault–Erdös' theorem about adding countable sets to N-sets. A similar result holds true for pseudo Dirichlet sets.

0. INTRODUCTION

Pseudo Dirichlet, N and N_0 are notions of smallness that arose in the theory of trigonometric series. $\mathfrak{m}$ and $\mathfrak{p}$ are cardinals between $\aleph_1$ and the continuum that play a role in many recent consistency and independence results in the set theory. The necessary definitions are given in Sections 1 and 2.

It turned out that the classical result of J.Arbault [Ar] saying that every countable subset of the unit interval is an N_0-set can be extended by replacing "countable" by "of cardinality smaller than $\mathfrak{m}$" (N. N. Kholshchevnikova [Kh]) or even by "of cardinality smaller than $\mathfrak{p}$" (Z.Bukovská [B1]). Another famous result about small sets of trigonometric series theory is the theorem of Arbault-Erdös that by adding a countable set to an N-set one obtains an N-set.[1] We show (see Theorem 1 below) that also in this case the word "countable" can be replaced by the words "of cardinality smaller than $\mathfrak{p}$". Moreover we show that a similar result for pseudo Dirichlet sets (Theorem 2 below) follows from theorems 3 and 10 of [B1].

Both J.Arbault [Ar] and N.Bary [Ba] ask whether the result about adding countable sets to an N-set can be extended for some uncountable set. The presented result shows that the question cannot be answered in a decisive way. We describe the answer to this question in two different models of the set theory ZFC.

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[1] J.Arbault in [Ar] proves this theorem with a remark that P.Erdös already has proved it independently without publishing. However, the notes of Arbault about Erdös' proof are sufficient for reconstructing it. The complete proof of P.Erdös is given in [Z,pp.237–238].

One can construct a model of ZFC in which the size $\mathfrak{c}$ of the continuum is any prescribed regular cardinal (say $\mathfrak{c} = \aleph_2$ or $\mathfrak{c} = \aleph_{\omega+1}$) and the so-called Martin Axiom holds true (see [Je,pp.232–241]). Consequently, in this model, $\mathfrak{p} = \mathfrak{c}$ and thus, by adding a set of cardinality smaller than $\mathfrak{c}$ (especially by adding a set of cardinality $\aleph_1$) to an N-set you obtain again an N-set.

On the other hand, one can construct a model of ZFC in which the size of the continuum is again arbitrarily large, but there exists a Lebesgue non-measurable set of cardinality $\aleph_1$ (see [Je,p.568]). Since every N-set has Lebesgue measure zero, this set is not an N-set and adding it to an N-set does not produce an N-set either.

1. Some set theory

We shall use the standard set-theoretic terminology and notation as introduced e.g. in [Fr,Je] (with slight differences, namely D.Fremlin uses inverse ordering of forcing).

By $|X|$ we denote the cardinality of the set X. $\mathbb{N}$ denotes the set of all natural numbers (including 0). If $\mathscr{A}$ is a family of subsets of a set A, then we define

$$\operatorname{non}(\mathscr{A}) = \min\{|X|;\ X \subseteq A\ \&\ X \notin \mathscr{A}\}.$$

A subset X of a partially ordered set $\mathscr{P}, \leq$ is said to be *centered* if for any finite system $x_1, \dots, x_n$ of elements of X, there is an $x \in \mathscr{P}$ such that $x \leq x_i$ for all $i = 1, \dots, n$. A partially ordered set $\mathscr{P}, \leq$ is *σ-centered* if $\mathscr{P}$ is a union of countably many centered subsets. A subset $\mathscr{A} \subseteq \mathscr{P}$ is *dense* in $\mathscr{P}$ if for every $x \in \mathscr{P}$ there exists a $y \leq x$, $y \in \mathscr{A}$. We shall suppose that the considered partially ordered sets do not have minimal elements. A subset $\mathscr{G}$ of $\mathscr{P}$ is called a *filter* if for any $x, y \in \mathscr{G}$ there exists a $z \in \mathscr{G}$ such that $z \leq x$, $z \leq y$, and, if $x \in \mathscr{G}$ and $x \leq z$, then also $z \in \mathscr{G}$.

The cardinal $\mathfrak{p}$ (see e.g. [vD,Fr]) is the minimum number of dense subsets in a σ-centered partially ordered set such that no filter meets them all. M. G. Bell [Be] provides a simpler, equivalent, combinatorial description of the cardinal $\mathfrak{p}$.

2. Some trigonometric series theory

Let us recall that a sequence of real-valued functions $\{f_n\}_{n=0}^{\infty}$ *quasinormally converges* to a function f on a set X if there exists a sequence of positive reals $\{\varepsilon_n\}_{n=0}^{\infty}$ converging to zero such that

$$(\forall x \in X)(\exists k)(\forall n > k) \mid f_n(x) - f(x) \mid \leq \varepsilon_n.$$

The quasinormal convergence was introduced and studied in [B2,CL] (Á.Császár and M.Laczkovich call it equal convergence).

We recall some notions and results of trigonometric series theory (for a recent survey of related notions and results see [BL,KS]). A subset A of the unit interval $[0, 1]$ is called a *Dirichlet set* (a *pseudo Dirichlet set*) if there exists an increasing sequence $\{n_k\}_{k=0}^{\infty}$ of natural numbers such that the sequence $\sin n_k \pi x$, $k = 0, 1, 2, \dots$, converges uniformly (quasinormally) to zero on A. The notion of a Dirichlet set is a rather classical one (see e.g. [KJ]). The notion of a pseudo Dirichlet set has been explicitly introduced in [B1] under the name D-set and then by S.Kahane [KS].

A subset A of the unit interval $[0, 1]$ is called an N-*set* if there are non-negative reals a_n, $n = 0, 1, 2, \ldots$, such that

$$\sum_{n=0}^{\infty} a_n = +\infty \tag{1}$$

and

$$\sum_{n=0}^{\infty} a_n \mid \sin \pi n x \mid < +\infty \quad \text{for every } x \in A. \tag{2}$$

Now, we use the ideas of the proof qiven by R.Salem [Sa] as modified by N.Bary. We follow N.Bary [Ba,Kh.XIII,§8]. Let A be an N-set, a_n, $n = 0, 1, \ldots$, being non-negative reals such that (1) and (2) hold true. We can assume that $0 < a_n \le 1$ for every n, $a_0 = 1$. Put

$$S_n = \sum_{i=0}^{n} a_i \tag{3}$$

and

$$\varrho_n = \frac{a_n}{S_n}. \tag{4}$$

Then the sequence $\{S_n\}_{n=0}^{\infty}$ is increasing and

$$\sum_{n=0}^{\infty} \varrho_n = +\infty. \tag{5}$$

Using some rather elementary facts from the infinite series theory, one can find an unbounded non-decreasing sequence $\{q_n\}_{n=0}^{\infty}$ of natural numbers such that

$$\sum_{n=0}^{\infty} \frac{a_n}{S_n^{1+\frac{1}{q_n}}} < +\infty. \tag{6}$$

Denote

$$\varepsilon_n = \frac{1}{S_n^{\frac{1}{q_n}}}. \tag{7}$$

We shall use these sequences in the proof of the main result of the paper.

The starting point for both proofs of Arbault-Erdös' theorem is a result of R.Salem [Sa] about adding a single point to an N-set, which is based on a classical result from the number theory.

Proposition 1. *For any reals* $x_1, \ldots, x_m$ *and any* $\varepsilon > 0$ *there exists a natural number* $k \neq 0$ *such that*

$$k \le \left(\frac{1}{\varepsilon}\right)^m \tag{8}$$

and

$$\mid \sin k\pi x_i \mid \le 2\pi\varepsilon \quad \textit{for } i = 1, \ldots, m. \tag{9}$$

The theorem is an easy consequence of the classical Dirichlet-Minkowski theorem about Diophantine approximations (see e.g.[Ba,B1,Ca]; note that $\mid \sin \pi x \mid$ is not greater than $\pi \times$ "the distance of x to the nearest integer").

Let us recall another well-known fact, which we shall need for the proof of Theorem 2.

Proposition 2. *If A is a Dirichlet set, $x \in [0, 1]$, then also $A \cup \{x\}$ is a Dirichlet set.*

A proof based on the Dirichlet-Minkowski theorem can be found e.g. in [Li].

3. Adding a set of cardinality $< \mathfrak{p}$ to an N-set

The following theorem is the main result of this paper.

Theorem 1. *Let $A \subseteq [0, 1]$ be an* N*-set. If $B \subseteq [0, 1]$ is a set of cardinality less than $\mathfrak{p}$, then $A \cup B$ is an* N*-set too.*

Before proving the theorem we introduce some notation and prove some auxiliary results.

Assume that A is an N-set and that $a_n, S_n, \varrho_n, q_n, \varepsilon_n$, $n = 0, 1, \dots$, are such as in Section 2 (i.e. conditions (1)–(7) hold true). For any finite subset T of B we denote

$$F_T = \{[n, k] \in \mathbb{N} \times \mathbb{N};\, 0 < k \leq S_n \ \& \\ \& \ |T| \leq q_n \ \& \ (|\sin kn\pi x| \leq 2\pi\varepsilon_n \text{ for every } x \in T)\}.$$

Let us remark that $F_T \subseteq F_S$ whenever $S \subseteq T$.

The following rather simple result expresses the main idea of Salem's proof [Sa].

Lemma 1. *If T is a finite subset of B, then for every n such that $q_n \geq |T|$ there exists an integer k such that $[n, k] \in F_T$.*

Proof. Assume that $|T| \leq q_n$. Then by Proposition 1, for $m = |T|$ and $\varepsilon = \varepsilon_n$ there exists a positive natural number $k \leq \left(\frac{1}{\varepsilon}\right)^m$ such that

$$|\sin kn\pi x| \leq 2\pi\varepsilon_n$$

holds true for every $x \in T$. By (7) and (8) we have

$$k \leq \left(\frac{1}{\varepsilon_n}\right)^m \leq \left(\frac{1}{\varepsilon_n}\right)^{q_n} = S_n. \quad \text{Q.E.D.}$$

The set-theoretical essence of Theorem 1 is hidden in the following lemma.

Main Lemma. *If $|B| < \mathfrak{p}$, then there exist a set $W \subseteq \mathbb{N}$ and a function λ defined on W with values positive integers such that*

$$(\forall k \in W)\lambda(k) \leq S_k, \tag{10}$$

$$\sum_{k \in W} \varrho_k = +\infty \tag{11}$$

and for every finite $T \subseteq B$ there is an m such that

$$[k, \lambda(k)] \in F_T \tag{12}$$

for every $k \in W$, $k \geq m$.

Proof. We define a partially ordered set $\mathscr{P}, \leq$ as follows (compare [Fr,pp.2–3] or [Vo,p.239]).

A couple $[t, T]$ belongs to $\mathscr{P}$ iff t is a function defined on a finite subset $\operatorname{dom}(t)$ of $\mathbb{N}$ with values positive integers, such that $t(k) \leq S_k$ for every $k \in \operatorname{dom}(t)$ and T is a finite subset of B. The order of $\mathscr{P}$ is defined by

$$[t_1, T_1] \leq [t_2, T_2] \equiv (\operatorname{dom}(t_2) \subseteq \operatorname{dom}(t_1) \;\&\; (\forall k \in \operatorname{dom}(t_2))\, t_1(k) = t_2(k) \\ \;\&\; (\forall k \in \operatorname{dom}(t_1) - \operatorname{dom}(t_2))\, [k, t_1(k)] \in F_{T_2} \;\&\; T_2 \subseteq T_1).$$

Since the set

$$\mathscr{P}_t = \{[t, T] \in \mathscr{P};\, T \text{ is a finite subset of } B\}$$

is centered, the set $\mathscr{P}$ being a countable union of $\mathscr{P}_t$'s is σ-centered. For a natural number m and a real $x \in B$ we denote

$$\mathscr{Q}_m = \{[t, T] \in \mathscr{P};\, \sum_{k \in \operatorname{dom}(t)} \varrho_k \geq m\},$$

$$\mathscr{R}_x = \{[t, T] \in \mathscr{P};\, x \in T\}.$$

It is easy to see that $\mathscr{Q}_m$ and $\mathscr{R}_x$ are dense in $\mathscr{P}$.

Since the set

$$\{\mathscr{Q}_m;\, m \in \mathbb{N}\} \cup \{\mathscr{R}_x;\, x \in B\}$$

has cardinality smaller than $\mathfrak{p}$, by the definition of $\mathfrak{p}$ there exists a filter $\mathscr{G} \subseteq \mathscr{P}$ that meets each $\mathscr{Q}_m$ and each $\mathscr{R}_x$.

We denote

$$W = \{k \in \mathbb{N};\, (\exists [t, T] \in \mathscr{G})\, k \in \operatorname{dom}(t)\}.$$

If $[t_1, T_1], [t_2, T_2] \in \mathscr{G}$, then there exists a couple $[s, S] \in \mathscr{G}$ such that $[s, S] \leq [t_i, T_i]$ for $i = 1, 2$. By the definition of the order $\leq$ we obtain that $t_1(k) = s(k) = t_2(k)$ for $k \in \operatorname{dom}(t_1) \cap \operatorname{dom}(t_2)$. Therefore, we can define the function λ as follows: for $k \in W$ we set

$$\lambda(k) = n \qquad \text{if } t(k) = n \text{ for some } [t, T] \in \mathscr{G}.$$

Evidently, for every $k \in W$ we have $\lambda(k) \leq S_k$. Thus (10) holds true.

By the choice of $\mathscr{G}$, for arbitrary $m \in \mathbb{N}$ we have $\mathscr{Q}_m \cap \mathscr{G} \neq \varnothing$. Therefore $\sum_{k \in W} \varrho_k \geq m$. As m was arbitrary, (11) holds true.

Now, let T be a finite subset of B. Since $\mathscr{R}_x \cap \mathscr{G} \neq \varnothing$ for every $x \in T$ and $\mathscr{G}$ is a filter, there exists a couple $[s, S] \in \mathscr{G}$ such that $T \subseteq S$. Let $m \in \mathbb{N}$ be greater than every $i \in \operatorname{dom}(s)$. If $k \geq m$, $k \in W$, then $k \in \operatorname{dom}(u)$ for some $[u, U] \in \mathscr{G}$, $[u, U] \leq [s, S]$. Then $\lambda(k) = u(k)$ and

$$[k, u(k)] \in F_S \subseteq F_T.$$

Thus, (12) holds true. Q.E.D.

Proof of Theorem 1. Let W, λ be as in the Main Lemma. We show that

$$\sum_{n \in W} \varrho_n \,|\sin n\lambda(n)\pi x\,| < +\infty \tag{13}$$

for every $x \in A \cup B$.

If $x \in A$, then

$$\varrho_n \,|\sin n\lambda(n)\pi x\,| \leq \varrho_n \lambda(n)\,|\sin n\pi x\,| \\ \leq \varrho_n S_n \,|\sin n\pi x\,| = a_n \,|\sin n\pi x\,|$$

for every $n \in W$. Thus, (13) is true by (2).

Now, let $x \in B$. By the Main Lemma there exists an integer m such that for every $n \geq m$, $n \in W$, we have

$$|\sin n\lambda(n)\pi x| \leq 2\pi\varepsilon_n.$$

Then

$$\varrho_n |\sin n\lambda(n)\pi x| \leq 2\pi \frac{a_n}{S_n^{1+\frac{1}{q_n}}}$$

and (13) holds true by (6).

The integers $n\lambda(n)$, $n \in W$, are not necessarily increasing, but only a finite number of them can be equal to any given integer. Rearranging the series (13) by setting

$$\varrho_k' = \sum\{\varrho_n ; n\lambda(n) = k\}$$

(and $\varrho_k' = 0$ if there is no n such that $n\lambda(n) = k$), we obtain the trigonometric series

$$\sum_{k=0}^{\infty} \varrho_k' \sin k\pi x$$

converging absolutely on $A \cup B$ with a divergent sum of coefficients. Q.E.D.

4. Adding a set of cardinality $< \mathfrak{p}$ to a pseudo Dirichlet set

We recall two results proved in [B1]. Theorem 3 of [B1] can be shortly formulated as follows.

Proposition 3. *A set is pseudo Dirichlet if and only if it is a union of an increasing sequence of Dirichlet sets.*

The next proposition is Theorem 10 of [B1].

Proposition 4. *Let $\{B_s, s \in S\}$ be a family of Dirichlet sets. If $|S| < \mathfrak{p}$ and for every finite $T \subseteq S$ the union $\bigcup_{s\in T} B_s$ is a Dirichlet set, then the union $\bigcup_{s\in S} B_s$ is a pseudo Dirichlet set.*

Using Propositions 2 – 4 we prove

Theorem 2. *If A is a pseudo Dirichlet set and $B \subseteq [0, 1]$ has cardinality less than $\mathfrak{p}$, then also $A \cup B$ is a pseudo Dirichlet set.*

Proof. Since A is a pseudo Dirichlet set, by Proposition 3 there are Dirichlet sets $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \subseteq \cdots$ such that $A = \bigcup_{n=0}^{\infty} A_n$. Denote by $\mathscr{F}$ the set of all finite subsets of B. For $n \in \mathbb{N}$ and $T \in \mathscr{F}$ we denote

$$B_{n,T} = A_n \cup T.$$

By Proposition 2, every $B_{n,T}$ is a Dirichlet set. One can easily see that the family $\{B_{n,T} ; [n, T] \in \mathbb{N} \times \mathscr{F}\}$ satisfies the assumption of Proposition 4. Therefore the union

$$\bigcup_{[n,T]\in\mathbb{N}\times\mathscr{F}} B_{n,T} = A \cup B$$

is a pseudo Dirichlet set. Q.E.D.

5. Some problems

According to what has been said in the introductory part, the question of N.Bary [Ba] "is there an uncountable set such that by adding it to any N-set one obtains again an N-set?" can be naturally modified as follows:

Problem 1. *What can you say about the smallest cardinal κ such that there are an N-set A and a set B of cardinality κ such that $A \cup B$ is not an N-set?*

Theorem 1 says that $\kappa \geq \mathfrak{p}$.

Let us denote by $\mathcal{N}$ the set of all N-subsets of the unit interval, and similarly $\mathcal{N}_0$ for an N_0-set (an N_0-set is an N-set with $a_n = 0, 1$; see e.g. [Ar,Ba,B1,BL, KS]) and $\mathcal{PD}$ for pseudo Dirichlet sets. By the result of [B1] mentioned in the introduction (which is also a simple consequence of Theorem 2) we have

$$\mathfrak{p} \leq \mathrm{non}(\mathcal{PD}) \leq \mathrm{non}(\mathcal{N}_0) \leq \mathrm{non}(\mathcal{N}).$$

Problem 2. *Can any of these inequalities be replaced by the equality?*

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Katedra matematickej analýzy, Prírodovedecká fakulta UPJŠ, Jesenná 5, 04154 Košice, Slovakia

Katedra matematickej informatiky, Prírodovedecká fakulta UPJŠ, Jesenná 5, 04154 Košice, Slovakia

E-mail address: `bukovsky@kosice.upjs.sk`