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SEMI-FREDHOLM OPERATORS WITH FINITE ASCENT OR DESCENT AND PERTURBATIONS

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ABSTRACT. In this note we prove that the collection of upper (lower) semi-Fredholm operators with finite ascent (descent) is closed under commuting operator perturbations that belong to the perturbation class associated with the set of upper (lower) semi-Fredholm operators. Then, as a corollary we get the main result of S. Grabiner (Proc. Amer. Math. Soc. **71** (1978), 79–80).

Let X be an infinite-dimensional complex Banach space and denote the set of bounded (compact) linear operators on X by $B(X)$ $(K(X))$. For T in $B(X)$ throughout this paper $N(T)$ and $R(T)$ will denote, respectively, the null space and the range space of T. Set $N^\infty(T) = \bigcup_n N(T^n)$, $R^\infty(T) = \bigcap_n R(T^n)$, $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim X/R(T)$. Recall that an operator $T \in B(X)$ is semi-Fredholm if $R(T)$ is closed and at least one of $\alpha(T)$ and $\beta(T)$ is finite. For such an operator we define an index $i(T)$ by $i(T) = \alpha(T) - \beta(T)$. Let $\Phi_+(X)$ $(\Phi_-(X))$ denote the set of upper (lower) semi-Fredholm operators, i.e., the set of semi-Fredholm operators with $\alpha(T) < \infty$ $(\beta(T) < \infty)$. The perturbation classes associated with $\Phi_+(X)$ and $\Phi_-(X)$ are denoted, respectively, by $P(\Phi_+(X))$ and $P(\Phi_-(X))$, i.e.,

$$P(\Phi_+(X)) = \{T \in B(X) \colon T + S \in \Phi_+(X) \text{ for all } S \in \Phi_+(X)\}$$

and

$$P(\Phi_-(X)) = \{T \in B(X) \colon T + S \in \Phi_-(X) \text{ for all } S \in \Phi_-(X)\}.$$

Recall that $a(T)$ $(d(T))$, the ascent (descent) of $T \in B(X)$, is the smallest non-negative integer n such that $N(T^n) = N(T^{n+1})$ $(R(T^n) = R(T^{n+1}))$. If no such n exists, then $a(T) = \infty$ $(d(T) = \infty)$. For a subset M of X let $\overline{M}$ denote the closure of M. The main result of this note is the following theorem.

Theorem 1. *Suppose that $T, K \in B(X)$ and $TK = KT$. Then*

$$(1.1) \qquad T \in \Phi_+(X),\ a(T) < \infty \text{ and } K \in P(\Phi_+(X)) \Rightarrow a(T+K) < \infty,$$

$$(1.2) \qquad T \in \Phi_-(X),\ a(T) < \infty \text{ and } K \in P(\Phi_-(X)) \Rightarrow d(T+K) < \infty.$$

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Proof. To prove (1.1) suppose that $T \in \Phi_+(X)$, $a(T) < \infty$, $K \in P(\Phi_+(X))$ and $TK = KT$. Set

$$T_\lambda = T + \lambda K, \qquad \lambda \in [0, 1].$$

For each $\lambda \in [0, 1]$, $T_\lambda \in \Phi_+(X)$. By [2, Theorem 3], there exists $\varepsilon = \varepsilon(\lambda) > 0$ such that

$$\overline{N^\infty(T_\lambda)} \cap R^\infty(T_\lambda) = \overline{N^\infty(T_\mu)} \cap R^\infty(T_\mu) \tag{1.3}$$

in the open disc $S(\lambda)$ with center λ and radius ε. Since $[0, 1]$ is compact, we can obtain a finite set $\{\lambda_0, \lambda_1, \dots, \lambda_n\}$ of points on $[0, 1]$ such that $\lambda_0 = 0$, $\lambda_n = 1$ and $[0, 1] \subset \bigcup_{i=0}^n S(\lambda_i)$ with $S(\lambda_i) \cap S(\lambda_{i+1}) \neq \emptyset$ for $i = 0, 1, \dots, n-1$. Now since $a(T) < \infty$, it follows that $N^\infty(T) \cap R^\infty(T) = \overline{N^\infty(T)} \cap R^\infty(T) = \{0\}$ [6, Proposition 1.6. (i)], and by (1.3) we have that $\overline{N^\infty(T_\mu)} \cap R^\infty(T_\mu) = \{0\}$ for all μ in $S(\lambda_0)$. Hence, because $S(\lambda_1)$ overlaps $S(\lambda_0)$, we conclude that $\overline{N^\infty(T_\mu)} \cap R^\infty(T_\mu) = \{0\}$ for all μ in $S(\lambda_1)$. By proceeding along the family of disc, we finally deduce that $\overline{N^\infty(T_\mu)} \cap R^\infty(T_\mu) = \{0\}$ for all μ in $S(\lambda_n)$. Thus $N^\infty(T_{\lambda_n}) \cap R^\infty(T_{\lambda_n}) = \{0\}$, and again by [6, Proposition 1.6. (i)] it follows that $a(T + K) < \infty$. This completes the proof of (1.1).

To prove (1.2) suppose that $T \in \Phi_-(X)$, $d(T) < \infty$, $K \in P(\Phi_-(X))$ and $TK = KT$. Then $T^* \in \Phi_+(X^*)$, $a(T^*) < \infty$, $T^*K^* = K^*T^*$ and $T^* + \lambda K^* \in \Phi_+(X^*)$, $\lambda \in [0, 1]$ [1, pp. 7–8]. Part (1.2) now follows directly from the proof of part (1.1). This completes the proof.

Let us remark that the commutativity condition in Theorem 1 is essential, even for compact K [1, pp. 13–14]. In order to prove Theorem 1 we need the hypothesis that K commutes with T in the place where we invoke [2, Theorem 3].

Now as a corollary, we get the main result of S. Grabiner [3, Theorem 2] (see also [4, Theorem 7.9.2]). Our formulation of that result is somehow different from that of S. Grabiner's, but appropriate to Theorem 1.

Corollary 2. *Suppose that $T \in B(X)$, $K \in K(X)$ and $TK = KT$. Then*

$$T \in \Phi_+(X) \text{ and } a(T) < \infty \Rightarrow a(T + K) < \infty, \tag{2.1}$$

$$T \in \Phi_-(X) \text{ and } d(T) < \infty \Rightarrow d(T + K) < \infty. \tag{2.2}$$

Proof. By Theorem 1 and the fact that $K(X) \subset P(\Phi_+(X)) \cap P(\Phi_-(X))$ [1, Theorem 5.6.9].

To help readers understand how far our Theorem 1 extends the result in [3], we refer them to the discussion of perturbation ideals in Sections 5.5 and 5.6, pages 95–102 in [1]. Let us mention in particular that $P(\Phi_+(X))$ includes all strictly singular operators.

Let $\sigma_a(T)$ and $\sigma_d(T)$ denote, respectively, the approximate point spectrum and the approximate defect spectrum of an element T of $B(X)$. Set

$$\sigma_{ab}(T) = \bigcap_{\substack{TK=KT\\ K\in K(X)}} \sigma_a(T + K) \quad \text{and} \quad \sigma_{db}(T) = \bigcap_{\substack{TK=KT\\ K\in K(X)}} \sigma_d(T + K).$$

We call $\sigma_{ab}(T)$ and $\sigma_{db}(T)$, respectively, Browder's essential approximate point spectrum of T and Browder's essential approximate defect spectrum of T [5], [7]. Recall that by [5, Theorem 2.1] a complex number $\lambda \notin \sigma_{ab}(T)$ $(\sigma_{db}(T))$

if and only if $T-\lambda \in \Phi_+(X)$, $i(T) \leq 0$ and $a(T-\lambda) < \infty$ $(T-\lambda \in \Phi_-(X)$, $i(T) \geq 0$ and $d(T-\lambda) < \infty)$.

Finally, as a second application of Theorem 1 we have

Corollary 3. *Suppose that* $T \in B(X)$. *Then*

$$\sigma_{ab}(T) = \bigcap_{\substack{TK=KT \\ K\in P(\Phi_+(X))}} \sigma_a(T+K) \tag{3.1}$$

and

$$\sigma_{db}(T) = \bigcap_{\substack{TK=KT \\ K\in P(\Phi_-(X))}} \sigma_d(T+K). \tag{3.2}$$

Proof. By Theorem 1 and [5, Theorem 2.1].

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