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COMPACTNESS CRITERIA FOR INTEGRAL OPERATORS IN L^∞ AND L^1 SPACES

S. P. EVESON

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ABSTRACT. Let (S, Σ, μ) be a positive measure space, $k: S \times S \to \mathbb{R}$ be a measurable function such that the kernel $|k|$ induces a bounded integral operator on $L^\infty(S, \Sigma, \mu)$ (equivalently, that $\operatorname{ess.sup}_{s \in S} |k(s, t)| \, d\mu(t) < \infty$), and for $s \in S$ let $k_s(t) = k(s, t)$. We show that it is sufficient for the integral operator T induced by k on $L^\infty(S, \Sigma, \mu)$ to be compact, that there exists a locally μ-null set $N \in \Sigma$ such that the set $\{k_s : s \in S\}$ is relatively compact in $L^1(S, \Sigma, \mu)$, and that this condition is also necessary if (S, Σ, μ) is separable. In the case of Lebesgue measure on a subset of $\mathbb{R}^n$, we use Riesz's characterisation of compact sets in $L^1(\mathbb{R}^n)$ to provide a more tractable form of this criterion.

1. INTRODUCTION

If (S, Σ, μ) is a positive measure space, $k: S \times S \to \mathbb{R}$ is a measurable function, and $p, q > 1$ with $p^{-1} + q^{-1} = 1$, one may define the 'double norm' of k on $L^p(S, \Sigma, \mu)$ by

$$\|k\|^p = \int_S \left\{ \int_S |k(s, t)|^q \, d\mu(t) \right\}^{p/q} d\mu(s).$$

If this double-norm is finite, then the integral operator induced by k is a compact transformation on $L^p(S, \Sigma, \mu)$ (see Zaanen [6, Chapter 11, §2, Example D]).

One may make the obvious generalisations of the double-norm to the spaces $L^1(S, \Sigma, \mu)$ and $L^\infty(S, \Sigma, \mu)$, but in these cases integral operators of finite double-norm, although bounded, are not necessarily compact. In the case of an integral operator T of finite double-norm on $L^1(S, \Sigma, \mu)$, T^2 turns out to be compact ([6, Chapter 11, §2, Example E]), but an operator of finite double-norm on $L^\infty(S, \Sigma, \mu)$ may fail even to be asymptotically compact ([6, Chapter 11, §2, Example D]).

We shall give necessary and sufficient conditions for an integral operator of finite double-norm on $L^\infty(S, \Sigma, \mu)$ to be compact, initially in the context of an abstract measure space and then in a more concrete form for operators on

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$L^\infty(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ with Lebesgue measure. Under these conditions, one also has that the transposed kernel induces a compact operator on $L^1(S, \Sigma, \mu)$.

2. Notation and hypotheses

The following notation and hypotheses will be used throughout.

Let (S, Σ, μ) be a positive measure space. We follow Hewitt and Stromberg [3] in our definition of the essential supremum and infimum and of $L^\infty(S, \Sigma, \mu)$. In brief, a set $N \in \Sigma$ is called *locally μ-null* if $\mu(A \cap N) = 0$ whenever $A \in \Sigma$ has finite measure. A measurable function $\phi\colon S \to \mathbb{R}$ is *essentially bounded above* if there is a locally μ-null set N and a real constant C such that $\phi(t) \le C$ for all $t \in S\backslash N$, in which case the essential supremum of ϕ is defined by

$$\operatorname{ess.sup}_S(\phi) = \inf\{C \in \mathbb{R}\colon \text{ for some locally } \mu\text{-null set } N,\ \phi(t) \le C \\ \text{for all } t \in S\backslash N\}.$$

The essential infimum is defined similarly, and $L^\infty(S, \Sigma, \mu)$ is then defined as the set of all measurable functions $\phi\colon (S, \Sigma, \mu) \to \mathbb{R}$ for which $|\phi|$ is essentially bounded above, quotiented by the subspace of such functions which are zero except on a locally null set. The norm is given by $\|\phi\|_\infty = \operatorname{ess.sup}_S(|\phi|)$.

Under these definitions, $L^\infty(S, \Sigma, \mu)$ may be identified in the usual way with the dual of $L^1(S, \Sigma, \mu)$, even if (S, Σ, μ) is not σ-finite. For $x \in L^1(S, \Sigma, \mu)$ and $\phi \in L^\infty(S, \Sigma, \mu)$ we let

$$\langle x, \phi\rangle = \int_S x\phi\, d\mu.$$

In the case where (S, Σ, μ) is σ-finite, these definitions are, of course, equivalent to the more conventional definitions where 'μ-null' is used in place of 'locally μ-null'.

We shall assume throughout that $k\colon S \times S \to \mathbb{R}$ is a measurable function. For $s \in S$ define $k_s\colon S \to \mathbb{R}$ by $k_s(t) = k(s, t)$. We shall assume that there exist a locally μ-null set $N \subseteq S$ and a constant $M > 0$ such that for $s \in S\backslash N$, $k_s \in L^1(S, \Sigma, \mu)$ and $\|k_s\|_1 \le M$. (The quantity $\operatorname{ess.sup}_{s\in S} \|k_s\|_1 \le M$ is the double-norm of k on $L^\infty(S, \Sigma, \mu)$ mentioned in the introduction.)

Define integral operators T_* and T by

$$(T\phi)(s) = \int_S k(s, t)\phi(t)\, d\mu(t),$$
$$(T_*x)(t) = \int_S k(s, t)x(s)\, d\mu(s).$$

Finally, let

$$P(S, \Sigma, \mu) = \left\{x \in L^1(S, \Sigma, \mu)\colon x \ge 0 \text{ almost everywhere and } \int_S x = 1\right\}.$$

3. Preliminaries

Definition 3.1. Let X be a real Banach space and $K \subseteq X$ be closed and convex. We shall call a set $\Phi \subseteq X^*$ *sufficient for* K if given any $x_0 \in X\backslash K$ there exist $\phi \in \Phi$ and $\alpha \in \mathbb{R}$ such that $\langle x_0, \phi\rangle < \alpha$ and $\phi > \alpha$ on K.

In this terminology, standard theory states that the entire dual space X^* is sufficient for any closed convex set K. In our application we shall, however, need *countable* sufficient sets. The next two lemmas show that such sets always exist, provided X is separable.

Lemma 3.1. *Let K be a closed convex subset with non-empty interior of a separable real Banach space X. Then there exists a countable set $\Phi \subseteq X^*$, sufficient for K.*

Proof. Let $\{y_n : n \in \mathbb{N}\}$ be a dense subset of $X\backslash K$. A standard separation theorem implies that for each $n \in \mathbb{N}$ there exists $\phi_n \in X^*$ and $\alpha_n \in \mathbb{R}$ such that $\langle y_n, \phi_n \rangle < \alpha_n$ and $\phi_n > \alpha_n$ on K. Let $\Phi = \{\phi_n : n \in \mathbb{N}\}$.

We now show that Φ has the required properties. Given $x_0 \in X\backslash K$, let $R = \operatorname{dist}(x_0, K)$ and choose $x_1 \in \overset{\circ}{K}$, so $B(x_1, \delta) \subseteq K$ for some $\delta > 0$.

Consider the balls $B_t = B(x_t, \delta t)$ for $0 < t < 1$, where $x_t = (1-t)x_0 + tx_1$ none of which contains the point x_0. Every point in B_t is at a distance at most $t(\|x_0 - x_1\| + \delta)$ from x_0, so by choosing $t < R/(\|x_0 - x_1\| + \delta)$ we have $B_t \cap K = \varnothing$. Since $\{y_n\}$ is dense, we can choose $y_n \in B_t$, so $\langle y_n, \phi_n \rangle < \alpha_n$ and $\phi_n > \alpha_n$ on K. It remains to be shown that $\langle x_0, \phi_n \rangle < \alpha_n$.

We have $y_n = x_t + v$, where $\|v\| < \delta t$. Let $x_2 = t^{-1}\{(x_t + v) - (1-t)x_0\}$. Now, $x_2 = x_1 + t^{-1}v$ and $\|t^{-1}v\| < \delta$, so $x_2 \in K$ and hence $\langle x_2, \phi_n \rangle > \alpha_n$. We also have $y_n = (1-t)x_0 + tx_2$, so $(1-t)x_0 = y_n - tx_2$. Hence, $(1-t)\langle x_0, \phi_n \rangle < \alpha_n - t\alpha_n$, so $\langle x_0, \phi_n \rangle < \alpha_n$. □

Theorem 3.1. *Let K be a closed convex subset of a separable real Banach space X. Then there exists a countable set $\Phi \subseteq X^*$, sufficient for K.*

Proof. For $n \in \mathbb{N}$, let $K_n = \{x \in X : \|x - y\| < n^{-1}$ for some $y \in K\}$. Since K_n is convex and has non-empty interior, by Lemma 3.1 there exists a countable set $\Phi_n \subseteq X^*$ such that for all $x_0 \in X\backslash\overline{K_n}$ there exists $\phi \in \Phi_n$ and $\alpha \in \mathbb{R}$ with $\langle x_0, \phi \rangle < \alpha$ and $\phi > \alpha$ on $\overline{K_n}$. Let $\Phi = \bigcup_{n \in \mathbb{N}} \Phi_n$.

To show that Φ has the required properties, pick $x_0 \in X\backslash K$ and let $R = \operatorname{dist}(x_0, K)$. Choose $n > R^{-1}$, so $x_0 \notin \overline{K_n}$. Now, there exist $\phi \in \Phi_n \subseteq \Phi$ and $\alpha \in \mathbb{R}$ such that $\langle x_0, \phi \rangle < \alpha$ and $\phi > \alpha$ on $\overline{K_n}$. Since $K \subseteq K_n$, $\phi > \alpha$ on K. □

In geometrical terms, this shows that a closed, convex subset of a separable Banach space can be represented as the intersection of a countable family of half-spaces.

Lemma 3.2. *Let A and B be subsets of a real Banach space X, and let $\Phi \subseteq X^*$ be sufficient for $\overline{\operatorname{co}}(B)$. Then $\overline{\operatorname{co}}(A) \subseteq \overline{\operatorname{co}}(B)$ if and only if for all $\phi \in \Phi$, $\overline{\operatorname{co}}(\langle A, \phi \rangle) \subseteq \overline{\operatorname{co}}(\langle B, \phi \rangle)$.*

Proof. Suppose $\overline{\operatorname{co}}(A) \subseteq \overline{\operatorname{co}}(B)$, and $\phi \in \Phi$. If $\phi = 0$, then $\overline{\operatorname{co}}(\langle A, \phi \rangle) = \overline{\operatorname{co}}(\langle B, \phi \rangle) = \{0\}$, otherwise pick $\varepsilon > 0$ and $t \in \operatorname{co}(\langle A, \phi \rangle)$, so $t = \langle x, \phi \rangle$ for some $x \in \operatorname{co}(A)$. Now, since $\operatorname{co}(A) \subseteq \overline{\operatorname{co}}(B)$, there exists $y \in \operatorname{co}(B)$ with $\|x - y\| < \varepsilon/\|\phi\|$, so $|\langle x, \phi \rangle - \langle y, \phi \rangle| < \varepsilon$, which is to say that $|t - \langle y, \phi \rangle| < \varepsilon$. Thus, $\operatorname{co}(\langle A, \phi \rangle) \subseteq \overline{\operatorname{co}}(\langle B, \phi \rangle)$.

Now suppose for all $\phi \in \Phi$, $\overline{\operatorname{co}}(\langle A, \phi \rangle) \subseteq \overline{\operatorname{co}}(\langle B, \phi \rangle)$, and assume for a contradiction that there exists $x_0 \in \overline{\operatorname{co}}(A)\backslash\overline{\operatorname{co}}(B)$. Since Φ is sufficient for $\overline{\operatorname{co}}(B)$, there exist $\phi \in \Phi$ and $\alpha \in \mathbb{R}$ such that $\langle x_0, \phi \rangle < \alpha$ and $\phi > \alpha$ on $\overline{\operatorname{co}}(B)$. Since ϕ is continuous, $\overline{\operatorname{co}}(\langle A, \phi \rangle)$ contains points less than α, but $\overline{\operatorname{co}}(\langle B, \phi \rangle)$ does not, contrary to the hypothesis. □

Lemma 3.3. *A linear map A on $L^1(S, \Sigma, \mu)$ is compact if and only if $A(P(S, \Sigma, \mu))$ is a relatively compact set.*

Proof. Suppose A is compact. Since $P(S, \Sigma, \mu)$ is bounded in the L^1 norm, $A(P(S, \Sigma, \mu))$ is relatively compact.

Now suppose that $A(P(S, \Sigma, \mu))$ is relatively compact and let B be the open unit ball in $L^1(S, \Sigma, \mu)$. For any $x \in B$, we have $x = \lambda u - \mu v$, where $u, v \in P(S, \Sigma, \mu)$ and $0 \leq \lambda, \mu \leq 1$. It now follows that $A(B)$ is relatively compact, so A is a compact operator. □

Lemma 3.4. *With the notation and hypotheses of Section 2, let $\phi \in L^\infty(S, \Sigma, \mu)$. Then*

$$\text{ess.}\sup_S(\phi) = \sup\left\{\int_S x\phi\, d\mu \colon x \in P(S, \Sigma, \mu)\right\},$$

$$\text{ess.}\inf_S(\phi) = \inf\left\{\int_S x\phi\, d\mu \colon x \in P(S, \Sigma, \mu)\right\}.$$

Proof. For $x \in P(S, \Sigma, \mu)$ we have

$$x(t)\,\text{ess.}\inf_S(\phi) \leq x(t)\phi(t) \leq x(t)\,\text{ess.}\sup_S(\phi)$$

for all t outside a locally μ-null set N. However, since $x \in L^1(S, \Sigma, \mu)$, the set $\{t \in S \colon x(t) \neq 0\}$ is σ-finite [3, Theorem 20.13], so this inequality holds for almost all t for which $x(t) \neq 0$. Thus,

$$\text{ess.}\inf_S(\phi) \leq \int_S x\phi\, d\mu \leq \text{ess.}\sup_S(\phi).$$

Given $\varepsilon > 0$, there exists a set E of finite positive measure on which $\phi > \text{ess.}\sup_S(\phi) - \varepsilon$. Let $x(t) = \chi_E(t)/\mu(E)$, so $x \in P(S, \Sigma, \mu)$ and

$$\int_S x\phi = (1/\mu(E)) \int_E \phi \geq \text{ess.}\sup_S(\phi) - \varepsilon.$$

Similarly, there exists $y \in P(S, \Sigma, \mu)$ such that $\int_S y\phi \leq \text{ess.}\inf_S(\phi) + \varepsilon$. □

4. Abstract measure spaces

The following result is an immediate consequence of the Fubini-Tonelli theorem.

Lemma 4.1. *The integral operators T and T_* defined in Section 2 are, under the hypotheses stated there, bounded operators on $L^\infty(S, \Sigma, \mu)$ and $L^1(S, \Sigma, \mu)$ respectively, and representing $L^1(S, \Sigma, \mu)^*$ by $L^\infty(S, \Sigma, \mu)$ in the usual way, T is the Banach space adjoint of T_*.*

We now give a generalisation of the following simple observation: if

$$P_n = \left\{x \in \mathbb{R}^n \colon x_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^n x_i = 1\right\}$$

and A is a real $n \times n$ matrix, then $A(P_n)$ is the convex hull of the columns of A.

Lemma 4.2. *With the notation and hypotheses of Section* 2,

$$\overline{\mathrm{co}}(\{k_s : s \in S\}) \supseteq \overline{T_*(P(S, \Sigma, \mu))}.$$

Proof. By Lemma 3.2 and the convexity of $T(P(S, \Sigma, \mu))$, it is sufficient to show that for each $\phi \in L^\infty(S, \Sigma, \mu)$,

$$\overline{\mathrm{co}}(\{\langle k_s, \phi\rangle : s \in S\}) \supseteq \overline{\langle T_*(P(S, \Sigma, \mu)), \phi\rangle}.$$

However,

$$\langle T_*(P(S, \Sigma, \mu)), \phi\rangle = \langle P(S, \Sigma, \mu), T\phi\rangle$$

and

$$\langle k_s, \phi\rangle = \int_S k(s, t)\phi(t)\, d\mu(t) = (T\phi)(s).$$

We thus need to show that

$$\overline{\mathrm{co}}(\{(T\phi)(s) : s \in S\}) \supseteq \overline{\langle P(S, \Sigma, \mu), T\phi\rangle}.$$

But by Lemma 3.4, $\overline{\langle P(S, \Sigma, \mu), T\phi\rangle} = [\text{ess.inf}_S(T\phi),\ \text{ess.sup}_S(T\phi)]$, and the result follows immediately. □

Lemma 4.3. *With the notation and hypotheses of Section* 2, *suppose* (S, Σ, μ) *is separable. Then there exists a locally* μ*-null set* $N \subseteq S$ *such that*

$$\overline{\mathrm{co}}(\{k_s : s \in S\backslash N\}) = \overline{T_*(P(S, \Sigma, \mu))}.$$

Proof. For any locally μ-null set N, if we choose $s_0 \in S\backslash N$ and define a kernel k' by

$$k'(s, t) = \begin{cases} k(s, t) & \text{if } s \notin N, \\ k(s_0, t) & \text{if } s \in N, \end{cases}$$

then the integral operator induced by k' is T_* and

$$\overline{\mathrm{co}}(\{k'_s : s \in S\}) = \overline{\mathrm{co}}(\{k_s : s \in S\backslash N\}).$$

Applying Lemma 4.2 to k' shows that

$$\overline{\mathrm{co}}(\{k_s : s \in S\backslash N\}) \supseteq \overline{T_*(P(S, \Sigma, \mu))}.$$

It remains to construct a locally μ-null set N for which the reverse inclusion is also true. Since $L^1(S, \Sigma, \mu)$ is separable, by Theorem 3.1 there exists a countable set $\Phi = \{\phi_n : n \in \mathbb{N}\} \subseteq L^\infty(S, \Sigma, \mu)$ which is sufficient for $\overline{T_*(P(S, \Sigma, \mu))}$.

Following the same reasoning as in Lemma 4.2, the reverse inclusion will be true if for all $n \in N$,

$$\overline{\mathrm{co}}(\{\langle k_s, \phi_n\rangle : s \in S\backslash N\}) = \overline{\mathrm{co}}(\{\langle T_* x, \phi_n\rangle : x \in P(S, \Sigma, \mu)\})$$

or, equivalently,

$$\overline{\mathrm{co}}(\{(T\phi_n)(s) : s \in S\backslash N\}) = \overline{\mathrm{co}}(\{\langle x, T\phi_n\rangle : x \in P(S, \Sigma, \mu)\}).$$

This, however, is equivalent by Lemma 3.4 to the two identities

$$\sup_{S\backslash N} T\phi_n = \text{ess.sup}_S\, T\phi_n,$$

$$\inf_{S\backslash N} T\phi_n = \text{ess.inf}_S\, T\phi_n.$$

Now, for $n \in \mathbb{N}$ let

$$N_n = \{s \in S\colon (T\phi_n)(s) > \operatorname{ess.sup}_S T\phi_n \text{ or } (T\phi_n)(s) < \operatorname{ess.inf}_S T\phi_n\}$$

and let $N = \bigcup_{n\in\mathbb{N}} N_n$.

For each $n \in \mathbb{N}$, we have $\phi_n(s) \le \operatorname{ess.sup}_S \phi_n$ for all $s \in S\backslash N$ and for any $\varepsilon > 0$, $\phi_n > \operatorname{ess.sup}_S \phi_n - \varepsilon$ on a set of positive measure in S, hence on a non-empty subset of $S\backslash N$. It follows that $\sup_{S\backslash N} T\phi_n = \operatorname{ess.sup}_S T\phi_n$; the other identity follows in a similar way. □

It is now easy to give necessary and sufficient conditions for T_* and T to be compact, based on the topology of the set of cross-sections $\{k_s\colon s \in S\}$ in $L^1(S, \Sigma, \mu)$.

Corollary 4.1. *With the notation and hypotheses of Section 2, if there exists a locally μ-null set $N \subseteq S$ such that the set $\{k_s\colon s \in S\backslash N\}$ is relatively compact in $L^1(S, \Sigma, \mu)$, then T_* and T are compact operators. If (S, Σ, μ) is separable, then the converse is also true.*

Proof. Pick $s_0 \in S\backslash N$ and define a kernel k' by

$$k'(s, t) = \begin{cases} k(s, t) & \text{if } s \notin N, \\ k(s_0, t) & \text{if } s \in N. \end{cases}$$

The integral operator induced by k' is T_* and

$$\overline{\operatorname{co}}(\{k'_s\colon s \in S\}) = \overline{\operatorname{co}}(\{k_s\colon s \in S\backslash N\}).$$

Applying Lemma 4.2 to k' shows that

$$\overline{\operatorname{co}}(\{k_s\colon s \in S\backslash N\}) \supseteq \overline{T_*(P(S, \Sigma, \mu))}.$$

Since $\{k_s\colon s \in S\backslash N\}$ is relatively compact in $L^1(S, \Sigma, \mu)$, its closed convex hull is compact. It follows that $T_*(P(S, \Sigma, \mu))$ is relatively compact, so T_* and T are compact by Lemma 3.3.

Now suppose that (S, Σ, μ) is separable and that T and T_* are compact (they are, of course, either both compact or both non-compact). Since $P(S, \Sigma, \mu)$ is bounded in $L^1(S, \Sigma, \mu)$, $\overline{T_*(P(S, \Sigma, \mu))}$ is a compact set. However, by Lemma 4.3, there exists a locally μ-null set N such that

$$\overline{\operatorname{co}}(\{k_s\colon s \in S\backslash N\}) = \overline{T_*(P(S, \Sigma, \mu))},$$

from which it follows that $\{k_s\colon s \in S\backslash N\}$ is relatively compact. □

5. Domains in $\mathbb{R}^n$

In the important special case that S is a subset of $\mathbb{R}^n$ and μ is Lebesgue measure, we can use a standard characterisation of L^1 compactness to provide alternative criteria. Lebesgue measure is, of course, both σ-finite and separable so 'locally null' reduces to 'null' and both implications of Corollary 4.1 are valid. We begin by recalling M. Riesz's characterisation of compact sets in $L^1(\mathbb{R}^n)$.

Theorem 5.1. *Let Ω be a (Lebesgue) measurable subset of $\mathbb{R}^n$ and K be a bounded subset of $L^1(\Omega)$. For $u \in K$, define $\tilde{u} \in L^1(\mathbb{R}^n)$ by*

$$\tilde{u}(x) = \begin{cases} u(t) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n\backslash\Omega. \end{cases}$$

Then K is relatively compact if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ and $R > 0$ such that for every $u \in K$ and for every $h \in \mathbb{R}^n$ with $|h| < \delta$,

$$\int_{\mathbb{R}^n \setminus B(0,R)} |\tilde{u}(x)|\,dx < \varepsilon, \qquad \int_{\mathbb{R}^n} |\tilde{u}(x+h) - \tilde{u}(x)|\,dx < \varepsilon.$$

Proof. See Riesz [4] or Adams [1, Theorem 2.21]. □

Corollary 5.1. *Let Ω be a measurable subset of $\mathbb{R}^n$ and $k\colon \Omega \times \Omega \to \mathbb{R}$ be a measurable function where there exists a constant $M > 0$ such that for almost all $x \in \Omega$, $k(x, \cdot) \in L^1(\Omega)$ and $\int_\Omega |k(x,y)|\,dy < M$. Define operators T and T_* on $L^\infty(\Omega)$ and $L^1(\Omega)$ respectively by*

$$(Tu)(x) = \int_\Omega k(x,y)u(y)\,dy,$$
$$(T_*v)(y) = \int_\Omega k(x,y)v(x)\,dx,$$

and define $\tilde{k}\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ by

$$\tilde{k}(x,y) = \begin{cases} k(x,y) & \text{if } y \in \Omega, \\ 0 & \text{if } y \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Then the following are equivalent:

(1) *T is compact.*

(2) *T_* is compact.*

(3) *Given $\varepsilon > 0$ there exist $\delta > 0$ and $R > 0$ such that for almost all $x \in \Omega$ and for every $h \in \mathbb{R}^n$ with $|h| < \delta$,*

$$\int_{\mathbb{R}^n \setminus B(0,R)} |\tilde{k}(x,y)|\,dy < \varepsilon, \qquad \int_{\mathbb{R}^n} |\tilde{k}(x,y+h) - \tilde{k}(x,y)|\,dy < \varepsilon.$$

Proof. Immediate from Corollary 4.1 and Theorem 5.1. □

6. Remarks

Although the results have been presented only for real L^1 and L^∞ spaces, their generalisations to complex spaces are immediate.

B. M. Cherkas [2] gives sufficient conditions for a subset of $L^\infty(S, \Sigma, \mu)$ to be compact, which are also necessary in the case that (S, Σ, μ) is σ-finite. It is easy to deduce from these that the conditions given in Corollary 4.1 are sufficient for T to be compact. Their necessity for separable measure spaces cannot, however, follow from Cherkas's criterion as stated in [2], since a separable measure space is not necessarily σ-finite.

The fact that the operator T_* is the integral operator whose kernel is the tranpose of that of T is not important; all that is required is the existence of a bounded operator T_* on $L^1(S, \Sigma, \mu)$ whose Banach space adjoint is T. It may be shown that such an operator exists if and only if T is bounded and weak $*$-continuous (this is Exercise 6 in Rudin [5, Chapter 4]), and all of the conclusions about T are valid in this case. It is not, however, clear what conditions other than finite double-norm could be placed on the kernel to guarantee such continuity of T.

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Department of Mathematics, University of York, Heslington, York, YO1 5DD, England

E-mail address: `spel@york.ac.uk`