PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 123, Number 12, December 1995

ON CHARACTERISTICS OF CIRCLE-INVARIANT PRESYMPLECTIC FORMS

AUGUSTIN BANYAGA AND PHILIPPE RUKIMBIRA

(Communicated by Linda Keen)

ABSTRACT. We prove that a circle-invariant exact 2-form of rank $2n$ on a compact $(2n+1)$-dimensional manifold admits two closed characteristics. This solves a particular case of a generalized Weinstein conjecture.

1. INTRODUCTION

A contact manifold is a couple (M, α), where M is a smooth manifold of dimension $2n+1 \geq 3$ and α is a contact form on M, that is, α is a 1-form such that $\alpha \wedge (d\alpha)^n$ is everywhere nonzero. The characteristic vector field of α is the unique vector field ξ on M such that $i_\xi \alpha = 1$ and $i_\xi d\alpha = 0$. Here, $i_{(.)}$ denotes the interior product operation. An integral curve of the vector field ξ is called a characteristic of the contact manifold (M, α).

In [WEI], Weinstein conjectured that there exists at least one closed characteristic on a compact (simply connected) contact manifold. Viterbo [VIT] proved this conjecture for compact contact manifolds (M, α) which are hypersurfaces of contact type in $\mathbf{R}^{2n+2}$; that is, if $\gamma: M \to \mathbf{R}^{2n+2}$ is the embedding of M into $\mathbf{R}^{2n+2}$ and $\Omega = \sum_{i=1}^{n+1} dx_i \wedge dy_i$ is the standard symplectic form, then $\gamma^*\Omega = d\alpha$. Several generalizations of the results were subsequently obtained by Hofer-Viterbo [HVI], [HOV]. Their methods are analytical: Viterbo first treated this problem as a problem of finding periodic solutions of Hamilton equations on an energy surface and translated it into finding critical points of some action functional on some loop space. The subsequent approach of Hofer and Viterbo uses, in addition, the theory of pseudo-holomorphic curves [HOV], [HVI], [HOF].

Recently, Hofer [HOF] proved the Weinstein conjecture for $\mathbf{S}^3$ and for any compact 3-manifold M with $\pi_2(M) \neq 0$ or the given contact form is overtwisted (see Eliashberg [ELI] for the definition). Observe that the closed characteristic Hofer finds must be a contractible loop.

Received by the editors June 14, 1994.
1991 *Mathematics Subject Classification.* Primary 58F05, 58F22; Secondary 53C15, 53C57.
Key words and phrases. Presymplectic form, characteristics, generalized Weinstein conjecture.
The first author was partially supported by NSF grant DMS 90-01861.

©1995 American Mathematical Society

In [BAN], one of us proved that if the contact manifold (M, α) is K-contact, then there are at least two closed characteristics and these are not necessarily contractible.

In this note, we show that this result is a consequence of a more general one on the existence of closed characteristics for exact circle-invariant 2-forms of maximum rank. In particular, the exterior derivative of a circle-invariant contact form falls into this category and therefore, we will obtain that any circle-invariant contact form on a compact manifold admits at least two closed characteristics. Moreover, our theorem guarantees the existence of two periodic solutions of the Hamilton equations on other hypersurfaces in $\mathbf{R}^{2n}$ which are not necessarily of contact type.

2. Characteristics of presymplectic forms and generalization of the Weinstein conjecture

Let M be a $(2n+p)$-dimensional manifold. A presymplectic form ω on M is a closed 2-form of constant rank $2n$. The kernel of ω is a completely integrable p-dimensional distribution and hence defines a p-dimensional foliation F_ω, called the characteristic foliation of ω.

Question. When does the foliation F_ω admit a compact leaf?

In this note we focus our attention on the case $\dim(F_\omega) = 1$. If ω is not exact, it is easy to construct, following Zehnder, a presymplectic form ω such that F_ω admits no compact leaf [ZEH].

We prove the following result:

Theorem. *Let M be a $(2n+1)$-dimensional oriented compact manifold with a 1-form α such that the 2-form $\omega = d\alpha$ has rank $2n$ everywhere. If there exists a locally free circle action on M which preserves ω, then the characteristic flow of ω has at least two closed leaves.*

Proof. For $s \in \mathbf{S}^1$, we denote by $s: M \to M$ the diffeomorphism given by the $\mathbf{S}^1$ action: $s(x) = s.x$. The 1-form

$$\alpha_0 = \int_{\mathbf{S}^1} (s^*\alpha)\, d\sigma$$

is $\mathbf{S}^1$-invariant. Here σ is the Haar measure on $\mathbf{S}^1$. Let Z be the infinitesimal generator of the $\mathbf{S}^1$ action. Then we have

$$L_Z\alpha_0 = 0,$$

where L_Z is the Lie derivative $L_Z(.) = i_Z d(.) + d i_Z(.)$. We define a smooth function $S: M \to R$ by

$$S(x) = -\alpha_0(x)(Z(x)),$$

that is, $S = -i_Z\alpha_0$. We have then $dS = i_Z d\alpha$. Indeed, observe first the fact that

$$\alpha_1 = \alpha - \alpha_0 = \int_{\mathbf{S}^1} (\alpha - s^*\alpha) d\sigma$$

is a closed 1-form because $s^* d\alpha = d\alpha$ and

$$d\alpha_1 = \int_{S^1} (d\alpha - s^* d\alpha) d\sigma = 0.$$

Hence

$$\begin{aligned} L_Z \alpha = L_Z(\alpha_0 + \alpha_1) &= L_Z \alpha_1 \quad \text{since } L_Z \alpha_0 = 0 \\ &= d i_Z \alpha_1 + i_Z d\alpha_1 = d i_Z \alpha_1 \quad \text{since } d\alpha_1 = 0. \end{aligned}$$

Let ξ be a nonsingular section of the characteristic distribution $F = F_\omega$ of ω. The vector field ξ verifies

$$i_\xi \omega = 0.$$

Hence, $L_\xi \omega = d i_\xi \omega + i_\xi d\omega = d i_\xi \omega = 0$ since $d\omega = 0$. We see that

$$i_{[\xi, Z]} \omega = L_\xi i_Z \omega - i_Z L_\xi \omega = 0 \tag{2.1}$$

since $L_\xi \omega = 0$ and

$$\begin{aligned} L_\xi i_Z \omega &= i_\xi d i_Z \omega + d i_\xi i_Z \omega \\ &= i_\xi d i_Z \omega = i_\xi d i_Z d\alpha = i_\xi d d S = 0. \end{aligned}$$

The identity (2.1) means that $[\xi, Z]$ is proportional to ξ, that is

$$[\xi, Z] = \lambda \xi$$

for some function λ. In other words, Z is a foliate vector field for the characteristic flow of $\omega = d\alpha$.

Let U be a flow box (for the flow of ξ) and

$$\psi : U \to D^{2n} \times (-\epsilon, \epsilon)$$

be a chart around $p \in U$ with $\psi(p) = ((0, 0), 0)$. For $q \in U$ we write

$$\psi(q) = ((x, y), z), \quad x = (x_1, \dots, x_n), \quad y = (y_1, \dots, y_n).$$

In this chart, $\xi_{|U} = \frac{\partial}{\partial z}$. The identity $[\xi, Z] = \lambda \xi$ then means that in our chosen coordinates, we have

$$Z(q) = \sum_{i=1}^n X_i(x, y) \frac{\partial}{\partial x_i} + \sum_{i=1}^n Y_i(x, y) \frac{\partial}{\partial y_i} + u(x, y, z) \frac{\partial}{\partial z}. \tag{2.2}$$

Now let $p \in M$ be a critical point of S. Since $i_Z d\alpha = i_Z \omega = dS$, one has

$$0 = (dS)(p) = \omega(p)(Z(p)).$$

This implies that

$$Z(p) = u(p) \xi(p)$$

since $\xi(p)$ spans the kernel of $\omega(p)$. Hence within our chart centered at $p \in U$, $\psi(p) = ((0, 0), 0)$, we have, by (2.2)

$$Z(p) = \sum_{i=1}^n X_i(0, 0) \frac{\partial}{\partial x_i} + \sum_{i=1}^n Y_i(0, 0) \frac{\partial}{\partial y_i} + u((0, 0), 0) \frac{\partial}{\partial z} = u((0, 0), 0) \frac{\partial}{\partial z}.$$

Let $q \in \mathscr{O}_p(\xi) \cap U$ where $\mathscr{O}_p(\xi)$ denotes the orbit of ξ through p. Then within our coordinates

$$\psi(q) = ((0, 0), t),$$

and

$$Z(q)=\sum_{i=1}^{n} X_i(0,0)\frac{\partial}{\partial x_i}+\sum_{i=1}^{n} Y_i(0,0)\frac{\partial}{\partial y_i}+u((0,0),t)\frac{\partial}{\partial z}=u(0,0,t)\frac{\partial}{\partial z}.$$

Hence Z is proportional to ξ at any point along the orbit of ξ in the flow box U. Covering $\mathcal{O}_p(\xi)$ by flow boxes, one sees that Z is proportional to ξ all along $\mathcal{O}_p(\xi)$, that is, Z is tangent to $\mathcal{O}_p(\xi)$. Therefore $\mathcal{O}_p(\xi)$ is an orbit of Z as well. Since all orbits of Z are closed, so is $\mathcal{O}_p(\xi)$.

We have just established that at a critical point p of S, the orbit of ξ passing through p is periodic. If S is not constant, it has at least two distinguished critical points, a maximum and a minimum since M is compact. These two points are on different orbits of ξ since S is constant along those orbits. To see this, we write

$$L_\xi S=i_\xi dS=i_\xi i_Z d\alpha=-i_Z i_\xi d\alpha=0.$$

Finally, if S is constant, then $dS=0$ so that

$$i_Z d\alpha=0,$$

which means that Z is everywhere proportional to ξ, hence all orbits of ξ are periodic. In any case, we see that there are at least two periodic orbits for ξ. □

Example 1. Let $(P, d\lambda)$ be an exact symplectic manifold with a circle action preserving the symplectic form $\Omega=d\lambda$. Let $H: P\to \mathbf{R}$ be a circle-invariant smooth function on P. Therefore any regular energy level $M=H^{-1}(a)$ (where $a\in\mathbf{R}$ is a regular value of H) is a C^∞ manifold equipped with a circle action which preserves the exact presymplectic form $\omega=\gamma^*\Omega$, where $\gamma: M\to P$ is the natural inclusion. Our theorem asserts that the hamiltonian vector field X_{H} admits two periodic orbits on the energy level M. For instance on $\mathbf{R}^{2n}$, the symplectic form $\Omega=\sum dx_i\wedge dy_i$ is invariant by $\mathbf{S}^1$. If $F:\mathbf{R}^{2n}\to\mathbf{R}$ is any smooth function and $H:\mathbf{R}^{2n}\to\mathbf{R}$ is defined as

$$H(x_1,\dots,x_n,y_1,\dots,y_n)=F(t_1,\dots,t_n)$$

where $t_i=x_i^2+y_i^2$, then H is $\mathbf{S}^1$-invariant. Therefore the hamiltonian equations

$$\dot{x}_i=-\frac{\partial H}{\partial y_i},\quad \dot{y}_i=\frac{\partial H}{\partial x_i}$$

admit at least two periodic solutions on each regular energy level.

Remark. We insist that in the example above, the hypersurface M is not necessarily of contact type.

3. Circle-invariant contact forms

As an immediate consequence of our theorem, we have the following

Corollary. *The characteristic vector field ξ_α of a compact contact manifold (M,α) where α is an $\mathbf{S}^1$-invariant contact form has at least two periodic orbits.*

Example 2. $\mathbf{S}^1$-invariant contact forms are well known in dimension 3, by Lutz's work [LUT]. On $\mathbf{S}^3$ for example, the exotic contact forms [BEN]

$$\alpha_n=\cos\left(\frac{\pi}{4}+n(x_3^2+x_4^2)\right)(x_1dx_2-x_2dx_1)+\sin\left(\frac{\pi}{4}+n(x_3^2+x_4^2)\right)(x_3dx_4-x_4dx_3)$$

are $\mathbf{S}^1$-invariant. As an immediate consequence of our theorem, all of those forms have at least two closed characteristics.

Also, the existence of at least two closed characteristics for a K-contact manifold [BAN] is a consequence of the following:

Proposition 1. *K-contact forms are circle-invariant.*

Proof. Let α be a K-contact form with characteristic vector field ξ and K-contact metric g. In particular, K-contactness requires that $L_\xi g = 0$ and $\alpha(X) = g(\xi, X)$ for all vector fields X [BLA]. By the classical theorem of Meyer-Steenrod, the group $I(M)$ of isometries of the compact riemannian manifold (M, g) is a compact Lie group. Let G be the closure in $I(M)$ of the 1-parameter group φ_t, $t \in \mathbf{R}$, generated by ξ. Then G is a compact abelian Lie group, hence a torus $\mathbf{T}^k = \mathbf{S}^1 \times \cdots \times \mathbf{S}^1$. Therefore $\mathbf{S}^1 \hookrightarrow \mathbf{T}^k$ acts smoothly on M. For any $s \in \mathbf{S}^1$, $s \circ \varphi_t = \varphi_t \circ s$ since both s and φ_t belong to the abelian group G. Hence $(s_*\xi) = \xi(s(x))$. Therefore

$$\begin{aligned} s^*\alpha(x)(X) &= \alpha(s(x))(s_*X) = g(s(x))(\xi(s(x)), s_*X)(x)) \\ &= g(s(x))(s_*\xi(x), (s_*X(x)) = s^*g(xi, X) \\ &= g(x)(\xi, X) = \alpha(X). \end{aligned}$$

Remark. We insist on the fact that an $\mathbf{S}^1$-invariant contact form is not necessarily K-contact. For example, the standard contact form

$$\alpha = \cos x_3 dx_1 + \sin x_3 dx_2$$

on the 3-dimensional torus is $\mathbf{S}^1$-invariant but cannot be K-contact [RUK].

For a given circle action on a contact manifold M, the class of circle-invariant contact forms is stable by multiplication by nowhere zero circle-invariant functions. Therefore our theorem leads to:

Proposition 2. *Let α_0 be a circle-invariant contact form on a compact manifold M. Let $u: M \to \mathbf{R}$ be a smooth positive function which is also invariant under the circle action. Then the charactersitic vector field ξ_u of $\alpha = u\alpha_0$ has at least two periodic orbits.*

Example 3. Let α_0 be the standard contact form on $\mathbf{S}^{2n+1}$ with characteristic vector field ξ_0, and let $u: \mathbf{S}^{2n+1} \to R$ be a positive function. Then

$$M = \{z.\sqrt{u(z)};\ z \in \mathbf{S}^{2n+1}\}$$

is a star-shaped hypersurface. Let X_M be its characteristic vector field (that is, the characteristic vector field of the restriction of $\sum dx_i \wedge dy_i$ to M). Bennequin [BEN1] has observed that the diffeomorphism

$$\psi: M \to \mathbf{S}^{2n+1}, \quad x \mapsto \frac{x}{\|x\|}$$

takes X_M to the characteristic vector field ξ_u of the contact form $\alpha = u\alpha_0$. If the function u is constant along the characteristics of α_0, then X_M has at least two periodic orbits. We already knew by Rabinowitz's theorem that X_M has one periodic orbit [RAB], for any smooth positive function u.

ACKNOWLEDGMENTS

The authors would like to thank A. Fathi for an important hint underlying this work. While this work was in gestation, the first author benefited from the hospitality of the ETH, Zurich (thanks to the invitation of J. Moser and E. Zehnder); the second author enjoyed the hospitality of the Mathematcs Department at Penn State University.

REFERENCES

[BAN] A. Banyaga, *A note on Weinstein conjecture*, Proc. Amer. Math. Soc. **109** (1990), 855–858.

[BAN1] ——, *On characteristics of hypersurfaces in symplectic manifolds*, Proc. Sympos. Pure Math., vol. 59, Part 2, Amer. Math. Soc., Providence, RI, 1993, pp. 4–17.

[BEN] D. Bennequin, *Entrelacements et équations de Pfaff*, Astérisque **107–108** (1983), 83–161.

[BEN1] ——, *Quelques remarques simples sur la rigidit é symplectique*, Géométrie Symplectique et de Contact: Autour du théorème de Poincaré-Birkhoff, Travaux en Cours (Dazord, Desolneux, and Moulis, eds.), Hermann, Paris, 1984, pp. 1–50.

[BLA] D. Blair, *Contact manifolds in riemannian geometry*, Lectures Notes in Math., vol 509, Springer, New York, 1976.

[ELI] Y. Eliashberg, *Contact 3-manifolds twenty years since J. Martinet's work*, Ann. Inst. Fourier (Grenoble) **42** (1992), 165–192.

[HVI] H. Hofer and C. Viterbo, *The Weinstein conjecture in the cotangent bundles and related results*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **15** (1988), 411–445.

[HOV] ——, *The Weinstein conjecture in the presence of holomorphic spheres*, Comm. Pure Appl. Math. **45** (1992), 583–622.

[HOF] H. Hofer, *Pseudoholomorphic curves in symplectizations with application to the Weinstein conjecture in dimension three*, preprint.

[LUT] R. Lutz, *Sur la géométrie des structures de contact invariantes*, Ann. Inst. Fourier (Grenoble) **29** (1979), 283–306.

[RAB] P. Rabinowitz, *Periodic solutions of hamiltonian systems*, Comm. Pure Appl. Math. **31** (1978), 157–184.

[RUK] P. Rukimbira, *Some remarks on R-contact flows*, Ann. Global Anal. Geom. **11** (1993), 165–171.

[VIT] C. Viterbo, *A proof of Weinstein conjecture in* $\mathbf{R}^{2n}$, Ann. Inst. H. Poincaré, Anal. Non Linéaire **4** (1987), 337–356.

[WEI] A. Weinstein, *On the hypothesis of Rabinowitz's periodic orbit theorem*, J. Differential Equations **33** (1978), 353–358.

[ZEH] E. Zehnder, *Remarks on periodic solutions on hypersurfaces*, Periodic Solutions of Hamiltonian Systems and Related Topics (P. Rabinowitz et al., eds.), Reidel, Dordrecht, 1987, pp. 267–279.

DEPARTMENT OF MATHEMATICS, PENN STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA 16802

E-mail address: `banyaga@math.psu.edu`

DEPARTMENT OF MATHEMATICS, FLORIDA INTERNATIONAL UNIVERSITY, MIAMI, FLORIDA 33199

E-mail address: `rukim@fiu.edu`