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HANDLEBODY ORBIFOLDS AND SCHOTTKY UNIFORMIZATIONS OF HYPERBOLIC 2-ORBIFOLDS

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ABSTRACT. The retrosection theorem says that any hyperbolic or Riemann surface can be uniformized by a Schottky group. We generalize this theorem to the case of hyperbolic 2-orbifolds by giving necessary and sufficient conditions for a hyperbolic 2-orbifold, in terms of its signature, to admit a uniformization by a Kleinian group which is a finite extension of a Schottky group. Equivalently, the conditions characterize those hyperbolic 2-orbifolds which occur as the boundary of a handlebody orbifold, that is, the quotient of a handlebody by a finite group action.

1. INTRODUCTION

Let F be a Fuchsian group, that is a discrete group of Möbius transformations—or equivalently, hyperbolic isometries—acting on the upper half space or the interior of the unit disk $\mathbf{H}^2$ (Poincaré model of the hyperbolic plane). We shall always assume that F has compact quotient $\mathscr{O} = \mathbf{H}^2/F$ which is a hyperbolic 2-orbifold with signature $(g; n_1, \dots, n_r)$; here g denotes the genus of the quotient which is a closed orientable surface, and $n_1, \dots, n_r$ are the orders of the branch points (the singular set of the orbifold) of the branched covering $\mathbf{H}^2 \to \mathbf{H}^2/F$. Then also F has signature $(g; n_1, \dots, n_r)$, and $(2 - 2g - \sum_{i=1}^r (1 - 1/n_i)) < 0$ (see [12] resp. [15] for information about orbifolds resp. Fuchsian groups). In case $r = 0$, or equivalently, if the Fuchsian group F is without torsion, the quotient $\mathbf{H}^2/F$ is a hyperbolic or Riemann surface without branch points. A classical uniformization theorem sometimes called the retrosection theorem says that any closed Riemann surface can be uniformized by a Schottky group (a special type of Kleinian group, i.e. a discrete group of Möbius transformations acting on the 2-sphere; see [6] for the theory of Kleinian groups). The main result of the present note is a generalization of this uniformization theorem to the case of hyperbolic 2-orbifolds, or equivalently, hyperbolic surfaces with branch or cone points. More precisely, we give necessary and sufficient conditions, in terms of its signature, such that a hyperbolic 2-orbifold $\mathscr{O} = \mathbf{H}^2/F$ can be uniformized by a Kleinian group E

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containing a Schottky group S as a subgroup of finite index (a finite extension of a Schottky group).

A *Schottky group* is a discrete group S of Möbius transformations of the 2-sphere $\mathbf{S}^2$ defined in the following way. Let $C_1, \dots, C_{2g}$ be simple closed curves on the 2-sphere which are the boundaries of $2g$ disjoint disks, and let $f_1, \dots, f_g$ be Möbius transformations such that f_i maps the interior of C_i to the exterior of C_{g+i}. Then the f_i generate a discrete group S of Möbius transformations (a Kleinian group), which is called a Schottky group. It is a free group, and the exterior of all $2g$ disks forms a fundamental domain for the action of S on the regular set $\Omega(S) = \mathbf{S}^2 - \Lambda(S)$ where $\Lambda(S)$ denotes the limit set of the Kleinian group S (which is a Cantor set in the case of a Schottky group of rank $g > 1$). The group S operates properly discontinuously and freely on $\Omega(S)$, and the quotient $\Omega(S)/S$ is a closed Riemann surface $\mathscr{F}_g$ of genus g. The retrosection theorem says that every Riemann surface can be obtained in this way. If $g > 1$, then the universal covering of the regular set $\Omega(S)$ is the hyperbolic plane $\mathbf{H}^2$ and the group consisting of all lifts of elements of S to $\mathbf{H}^2$ is a Fuchsian group F without torsion. In particular, $\mathscr{F}_g = \Omega(S)/S = \mathbf{H}^2/F$ becomes also a hyperbolic surface in a canonical way.

Now let F be an arbitrary Fuchsian group with compact quotient. We say that the hyperbolic 2-orbifold $\mathscr{O} = \mathbf{H}^2/F$ can be uniformized by a finite extension E of a Schottky group if the Kleinian group E contains a Schottky group S as a subgroup of finite index (in particular, $\Omega(E) = \Omega(S)$), and if the hyperbolic quotient orbifold $\Omega(E)/E$ is isometric to the hyperbolic orbifold $\mathscr{O} = \mathbf{H}^2/F$, or equivalently, if the Fuchsian group F is the lift of the group E from $\Omega(E)$ to the hyperbolic plane (up to conjugation by isometries).

Our main result is the following

Theorem 1. *Let $(g; n_1, \dots, n_r)$ be the signature of a closed orientable hyperbolic 2-orbifold $\mathscr{O}$. Then $\mathscr{O}$ can be uniformized by a finite extension E of a Schottky group, i.e. $\mathscr{O} = \Omega(E)/E$, if and only if one of the following three conditions holds.*

(i) $g > 0$;

(ii) *the (unordered) numbers $n_1, \dots, n_r$ occur in pairs;*

(iii) *among the numbers $n_1, \dots, n_r$ there are at least two disjoint occurrences out of the following four situations:* (2), (3, 3), (3, 4) *or* (3, 5) (*possibly two times the same*).

The action of any Kleinian group on the 2-sphere extends to an action on the 3-ball $\mathbf{B}^3$, properly discontinuous and by isometries on its interior which is a model of hyperbolic 3-space $\mathbf{H}^3$. For a Schottky group S, the quotient $(\mathbf{B}^3 - \Lambda(S))/S$ is a 3-dimensional handlebody $\mathscr{V}_g$ of genus g, with $\mathscr{F}_g$ as boundary. As a consequence, Theorem 1 has an interpretation in terms of extensions of finite group actions from surfaces to handlebodies. Let $\mathscr{F}_g$ be a closed orientable surface of genus $g > 1$ and G a finite group of orientation-preserving homeomorphisms of $\mathscr{F}_g$. It is no loss of generality to assume that $\mathscr{F}_g$ is a hyperbolic surface and that G acts by isometries. Then $\mathscr{O} := \mathscr{F}_g/G$ is a hyperbolic 2-orbifold and we have the following

Proposition. *Suppose the G-action on $\mathscr{F}_g$ extends to a handlebody $\mathscr{V}_g$. Then the signature of the quotient orbifold $\mathscr{O} = \mathscr{F}_g/G$ is of one of the three types given in the theorem.*

The necessary condition in the proposition for a finite group action on a surface to extend to a handlebody is far from being also sufficient. In fact, the condition gives only a rough first approximation to the problem which seems quite intractable in general because it depends too much on the structure of the finite group G and the specific action (see also [1]). However, for some classes of finite groups, e.g. cyclic, dihedral or abelian groups, it is easy to obtain necessary and sufficient conditions.

The conditions in Theorem 1 characterize also those hyperbolic 2-orbifolds which can be obtained as the boundary of a handlebody orbifold; see the next section. At the end of the next section we shall characterize those hyperbolic 2-orbifolds which occur as the boundary of infinitely many different resp. exactly one handlebody orbifold (Theorem 2).

2. Proofs of Theorem 1 and the Proposition; Theorem 2

Assuming Theorem 1 we will give first the

Proof of the Proposition. Let G be a finite group of orientation-preserving homeomorphisms of the closed orientable surface $\mathcal{F}_g$. Each nontrivial element of G has only isolated fixed points and operates as a rotation in a neighbourhood of each fixed point (see [3, pages 87 and 224]). It follows that the quotient $\mathcal{F}_g/G$ is again a closed orientable surface with a finite number of branch points, i.e. a 2-orbifold $\mathcal{O}$ of some signature $(g; n_1, \dots, n_r)$. This 2-orbifold can be uniformized by choosing some Fuchsian group of the same signature and so becomes a hyperbolic 2-orbifold. Lifting the hyperbolic structure to the surface $\mathcal{F}_g$ this becomes a hyperbolic or Riemann surface such that G acts by isometries resp. conformal automorphisms.

Now by hypothesis the G-action on $\mathcal{F}_g$ extends to a handlebody $\mathcal{V}_g$. By the precise version of the retrosection theorem, also called *theorem on cuts*, as proved by Koebe (see for example [4, page 35]), there exists a uniformization of $\partial\mathcal{V}_g = \mathcal{F}_g$ by a Schottky group S which uniformizes also the handlebody $\mathcal{V}_g$. Denote by $\Omega_e(S) := \mathbf{B}^3 - \Lambda(S)$ the regular set of S in the 3-ball $\mathbf{B}^3$. Then the elements of the group G lift to the universal covering $\Omega_e(S) \subset \mathbf{B}^3$ of $\mathcal{V}_g$, and the group E generated by all such lifts acts by conformal maps on $\Omega(S) = \Omega_e(S) \cap \mathbf{S}^2 \subset \mathbf{S}^2$. By Gehring's extension theorem, the action of E extends to a conformal action on the whole 2-sphere $\mathbf{S}^2$ (see [5, p. 281]). Therefore E is a Kleinian group which is a finite extension of the Schottky group S (the universal covering group of $\mathcal{V}_g$) and uniformizes the 2-orbifold $\mathcal{O} = \mathcal{F}_g/G = \Omega(E)/E$. Now by Theorem 1, the signature of $\mathcal{O}$ is of one of the 3 types given in the theorem.

Proof of Theorem 1. Suppose the Kleinian group E contains the Schottky group S as a subgroup of finite index and uniformizes the hyperbolic 2-orbifold $\mathcal{O} = \mathbf{H}^2/F$, for a Fuchsian group F, i.e. $\mathcal{O} = \Omega(E)/E$. By taking the intersection of S with its finitely many conjugates in E, we can assume that S is a normal subgroup of E. Now $\Omega_e(S) = \Omega_e(E)$ is invariant under the action of E, therefore we have an induced action of the finite group $G := E/S$ on the handlebody $\mathcal{V}_g := \Omega_e(S)/S$. Let D be a 2-dimensional properly embedded disk in $\mathcal{V}_g$ such that $\partial D = D \cap \partial\mathcal{V}_g$ is a nontrivial closed curve on $\mathcal{F}_g = \partial\mathcal{V}_g$. By the equivariant loop theorem/Dehn lemma ([7]), we can assume that $g(D) = D$

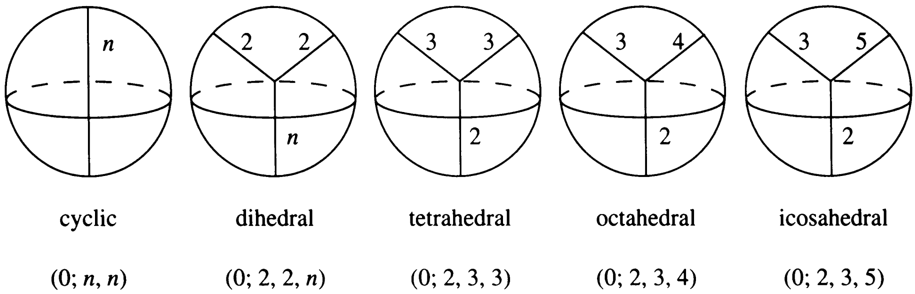

FIGURE 1

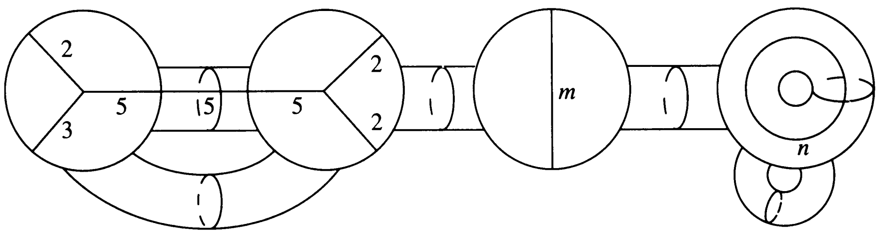

FIGURE 2

or $g(D) \cap D = \varnothing$, for all $g \in G$. When cutting $\mathscr{V}_g$ along the system of disjoint disks $G(D)$, that is, removing the interior of a G-invariant regular neighbourhood of $G(D)$ (which is a collection of 1-handles, that is products of a 2-disk with an interval), we get again a collection of handlebodies of lower genus on which G acts. Applying inductively the above procedure of cutting along disks, we finally end up with a collection of disjoint 3-balls on which G acts. Thus the quotient-orbifold $\mathscr{H} := \mathscr{V}_g/G$ is built up from orbifolds which are quotients of 3-balls by finite groups of homeomorphisms (their stabilizers in G), connected by finite cyclic quotients of 1-handles, which are the projections of the removed regular neighbourhoods of the disks (first type of orbifold in Figure 1). The finite orientation-preserving groups which can act on the 3-ball or the 2-sphere are the finite subgroups of the orthogonal group $SO(4)$. It is well known that, on the boundary of the 3-ball, the actions are standard, i.e. conjugate to orthogonal actions. By Thurston's orbifold geometrization theorem [13] (which is not needed for the proof of the theorem, however) the same is true for the whole 3-ball. The figures of the possible quotient-orbifolds, together with the signatures of the boundaries, are listed in Figure 1; the underlying topological space is the 3-ball in each case.

These quotient orbifolds are connected by the 1-handle orbifolds; the result is called a handlebody orbifold in [8]. An example of a handlebody orbifold is given in Figure 2; in this example the boundary has signature

$$(3; 2, 3, 2, 2, m, m).$$

To each handlebody orbifold $\mathscr{H}$ one can associate a graph of groups: the vertices resp. edges correspond to the quotients of the 3-balls resp. 1-handles, and to each vertex resp. edge we associate the corresponding finite group, writing only the orders of the non-trivial cyclic groups associated to the edges. Note

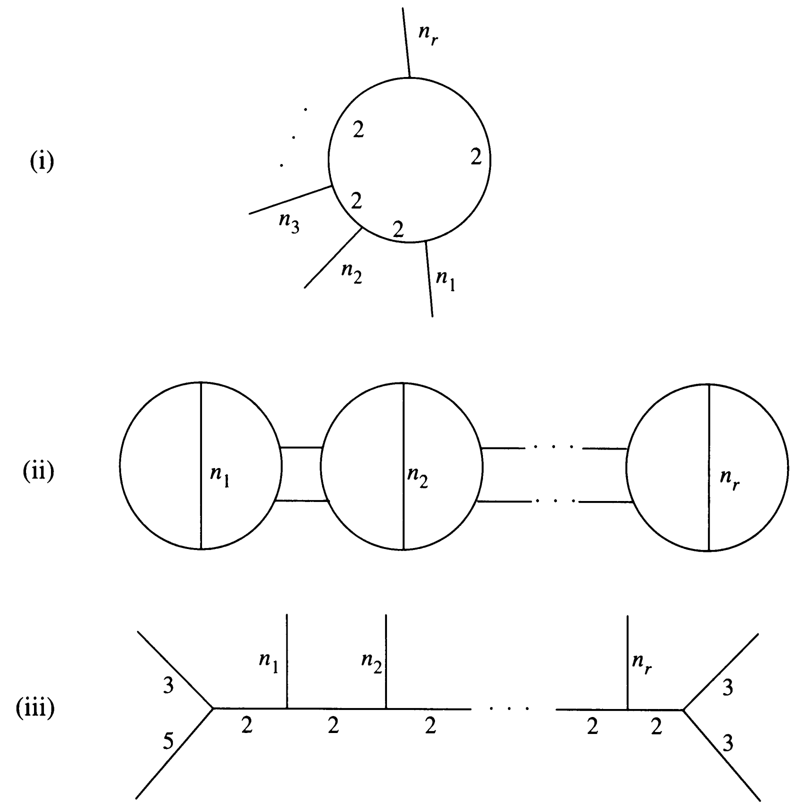

FIGURE 3

that, by the orbifold version of Van Kampen's theorem (see [2]), the orbifold fundamental group of $\mathscr{H}$, and therefore also the group E which is its universal covering group, is isomorphic to the fundamental group of the above graph of groups (see [11] or [14] for definitions about graphs of groups). Deleting the edges whose associated groups are trivial we get exactly the singular set of the handlebody orbifold $\mathscr{H}$. This singular set is a graph Γ all of whose vertices have valence 2 or 3. If some component of Γ is not simply connected, i.e. not a tree, then obviously the genus g of the 2-orbifold $\mathscr{O} = \partial\mathscr{H}$ is at least 1 so we are in case (i) of the theorem. Otherwise Γ consists of trees which are subdivided segments or contain 3-valent points. If we have only segments we are in case (ii) of the theorem; if there is at least one vertex of valence 3 we are in case (iii). This proves one direction of the theorem.

Now suppose that the hyperbolic 2-orbifold $\mathscr{O}$ of signature $(g, n_1, \dots, n_r)$ satisfies one of the three conditions (i), (ii) or (iii) of the theorem. In each of the three cases, it is possible, as indicated in Figure 3 (where we give only the structure of the singular set of the orbifold, except in case (ii)), to construct a handlebody orbifold $\mathscr{H}$ such that $\partial\mathscr{H} = \mathscr{O}$.

As above, the orbifold fundamental group $\pi_1(\mathscr{H})$ is the fundamental group of a finite graph of finite groups. By [10, Lemma 7.4] (see also [14, Proposition

2.1.1]), there exists a surjection of $\pi_1(\mathscr{H})$ onto a finite group G whose kernel is a free group. The regular covering of $\mathscr{H}$ corresponding to the kernel of this surjection is a handlebody $\mathscr{V}_g$ on which G acts as the group of covering transformations (see also [8, pp. 390/391]). The boundary of $\mathscr{V}_g$ is a finite covering of the hyperbolic 2-orbifold $\mathscr{O}$, so it becomes a hyperbolic or Riemann surface $\mathscr{F}_g$ on which G acts by isometries resp. conformal maps. Now, as in the proof of the proposition, there exists a finite extension E of a Schottky group S uniformizing the hyperbolic 2-orbifold $\mathscr{O} = \mathscr{F}_g/G = \Omega(E)/E$. (Alternatively, in order to construct the Kleinian group E, one may inductively apply combination theorems as in [9] where a more general situation has been considered.) This finishes the proof of Theorem 1.

We have seen in the proof of Theorem 1 that a hyperbolic 2-orbifold $\mathscr{O}$ can be uniformized by a finite extension E of a Schottky group if and only if there exists a handlebody orbifold $\mathscr{H}$ such that $\partial\mathscr{H} = \mathscr{O}$. In the next theorem we consider the following question: when is $\mathscr{H}$ unique (up to homeomorphisms); in general, how many handlebody orbifolds $\mathscr{H}$ exist such that $\partial\mathscr{H} = \mathscr{O}$?

Theorem 2. (a) *Let* $(g; n_1, \dots, n_r)$ *be the signature of a closed orientable hyperbolic 2-orbifold* $\mathscr{O}$. *There exist infinitely many handlebody orbifolds* $\mathscr{H}$ *such that* $\partial\mathscr{H} \simeq \mathscr{O}$ *if and only if the signature of* $\mathscr{O}$ *satisfies one of the following conditions*:

(i) $g \geq 2$;
(ii) $g = 1$ *and among the numbers* $n_1, \dots, n_r$ *there are at least two disjoint occurrences out of the following four situations*: (2), $(3, 3)$, $(3, 4)$ *or* $(3, 5)$;
(iii) $g = 0$ *and among the numbers* $n_1, \dots, n_r$ *there are at least four disjoint occurrences as above.*

(b) *There exists exactly one handlebody orbifold* $\mathscr{H}$ *such that* $\partial\mathscr{H} \simeq \mathscr{O}$ *if and only if the signature of* $\mathscr{O}$ *satisfies one of the following conditions*:

(i) $(0; 2, 2, a, b)$, $a \geq 2$, $b \geq 6$ *and* $a \neq b$;
(ii) $(0; 2, 3, a, a)$; $(0; 3, 3, a, a, a)$, $a = 3, 4, 5$;
(iii) $(0; 3, 3, a, a, b, c)$, $a = 3, 4, 5$, $b = a$ *or for* $b \geq 6$, $c \geq 6$ *and* $b \neq c$;
(iv) $g = 0$ *and among the unordered numbers* $n_1, \dots, n_r$ *there are exactly two disjoint occurrences out of the situations above and one more number which is* ≥ 6;
(v) $(1; a)$ *for* $a \neq 2$;
(vi) $(1; a, b)$ *for* $b \geq 6$ *and* $a \neq b$;
(vii) $(1; a, b, c)$ *where* a, b, c *are pairwise different and* $b \geq 6$, $c \geq 6$.

Proof. The proof is similar to the proof of Theorem 1, so we don't give the details. In cases (i) and (ii) of part (a) the singular sets of the infinitely many orbifolds $\mathscr{H}$ are constructed by adding a circle of an arbitrary branching order $n \geq 2$ to the singular sets shown in Figure 3, cases (i) and (iii) (see also Figure 2). For the case (iii), we join two singularity graphs as in Figure 3, case (iii), by an edge of arbitrary order $n \geq 2$ as shown in Figure 4. For (b) let us consider the case $g = 0$ (the case $g = 1$ is analogous). If there exist three disjoint occurrences out of the situations (2), (3,3), (3,4) or (3,5), then we can construct at least two singular sets as in Figure 3, case (iii), by different permutations of

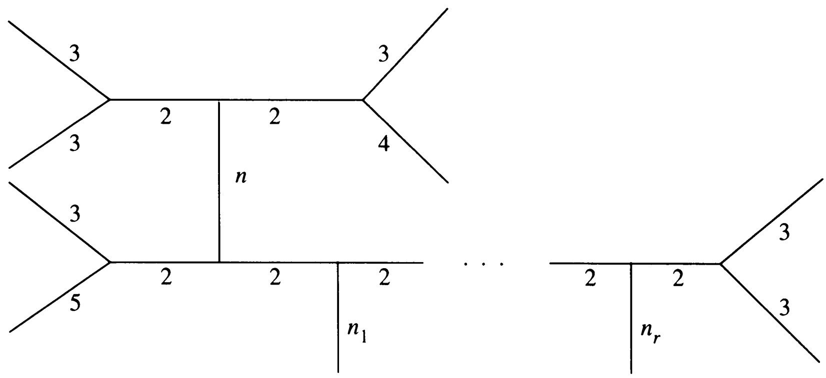

FIGURE 4

the n_i's (with the exception of the signature $(0; 2, 2, 2, b)$ for $b \geq 6$ listed in (b), case (i)). Suppose then that there are two disjoint occurrences out of the situations above and two more numbers n_i and n_j. If $n_i = n_j$, then we can construct a singular set as in Figure 3, case (iii), but also a disconnected singular set with one component as in Figure 3, case (iii) and another component as in Figure 3, case (ii). If $n_i \neq n_j$, then we can construct two different singular sets as in Figure 3, case (iii), by different permutations of the n_i's (with the exception of the signatures listed in (b), case (iii)). We are left with the case $r = 5$ which gives the signatures listed in (b), case (ii) and (b), case (iv).

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