ANY BEHAVIOUR OF THE MITCHELL ORDERING OF NORMAL MEASURES IS POSSIBLE

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ABSTRACT. Let U_0, U_1 be two normal measures on κ . We say that U_0 is in the Mitchell ordering less than U_1 , $U_0 \triangleleft U_1$, if $U_0 \in \text{Ult}(V, U_1)$. The relation is well-known to be transitive and well-founded. It has been an open problem to find a model where $\triangleleft$ embeds the four-element poset $\begin{smallmatrix} \circ & \circ \\ \parallel & \parallel \\ \circ & \circ \end{smallmatrix}$. We find a generic extension where all well-founded posets are embeddable. Hence there is no structural restriction on the Mitchell ordering. Moreover we show that it is possible to have two $\triangleleft$ -incomparable measures that extend in a generic extension into two $\triangleleft$ -comparable measures.

We address the question of possible behaviours of the Mitchell ordering of normal measures.

In the well-known Mitchell's model $L[\vec{U}]$ the ordering of measures on a cardinal κ is linear [Mi83]. S. Baldwin constructed a model where $\triangleleft$ is a pre-well-ordering [Ba85] (a well-founded poset is *pre-well-ordered* iff $\forall p, q \in P : p <_P q$ iff $o_P(p) < o_P(q)$ where $o_P(p)$ is the rank of p in P). Recently J. Cummings [Cu93] described the Mitchell ordering in a particular generic extension where it embeds any well-founded poset that does not embed the four-element poset:

We say that a well-founded poset P *embeds into the Mitchell ordering of normal measures on κ* if there are different measures $\{U_p; p \in P\}$ on κ so that $U_p \triangleleft U_q$ iff $p <_P q$. I show that there is a generic extension where any well-founded poset (with certain cardinality restrictions) embeds into $\triangleleft$. It proves that there is no structural restriction on the Mitchell ordering. However it still remains open to construct a model where all measures on κ ordered by the Mitchell ordering are isomorphic to a given poset, e.g. there are only four measures ordered according to the figure above. That would certainly need to go into inner models, possibly

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first starting with a generic extension. S. Baldwin in [Ba85] notes that his method hopelessly fails especially for posets that embed the four-element poset.

We say that a function $f : \kappa \rightarrow V_\kappa$ is a *Laver's function on κ* if

$$\forall x \in \mathcal{P}(\kappa^+) \exists U \text{ a measure on } \kappa : (j_U f)(\kappa) = x$$

where $j_U : V \rightarrow M_U = \text{Ult}(V, U)$ is the canonical ultraproduct embedding. It is proved in [W94a] that there is a Laver's function on κ if $V = L[\vec{U}]$ and $\mathcal{o}^{\mathcal{U}}(\kappa) = \kappa^{++}$. It follows from the fact that the measures on κ cover $\mathcal{P}(\kappa^+)$ in the following sense

$$\forall x \in \mathcal{P}(\kappa^+) \exists \alpha < \kappa^{++} : x \in \text{Ult}(V, U_\alpha^\kappa).$$

Theorem 1. *Assume that $V = L[\vec{U}]$, $\mathcal{o}^{\mathcal{U}}(\kappa) = \kappa^{++}$. Then there is a generic extension $V[G]$ preserving cardinalities, cofinalities, and GCH such that any well-founded poset in $V[G]$ of cardinality $\leq \kappa^+$ embeds into $\triangleleft_\kappa^{V[G]}$. Moreover there is still a Laver's function on κ .*

Let me first state a few well-known facts (see e.g. [Cu93] or [W94a]).

Fact 1. *Let $U \triangleleft W$ be two measures on κ , $j_U : V \rightarrow M_U$, $j_W : V \rightarrow M_W$, and $j_U^{M_W} : M_W \rightarrow M'_U = \text{Ult}(M_W, U)$. Then $j_U^{M_W} = j_U \restriction M_W$ and $V_{j_U(\kappa)+1} \cap M'_U = V_{j_U(\kappa)+1} \cap M_U$.*

Fact 2. *Let M, N be two inner models of ZFC, $M \subseteq N$, $N \cap {}^\kappa M \subseteq M$. Let $P \in M$ be a forcing notion such that $N \models \text{"}P \text{ is } \kappa^+\text{-c.c. or } P \text{ is } \kappa\text{-closed"}$, and let G be P -generic over N . Then $N[G] \cap {}^\kappa M[G] \subseteq M[G]$.*

Fact 3 (Easton's lemma). *Let a notion of forcing P be κ -c.c., and let Q be $< \kappa$ -closed. Then*

- (i) $Q \Vdash \text{"}\check{P} \text{ is } \check{\kappa}\text{-c.c.}"$,
- (ii) $P \Vdash \text{"}\check{Q} \text{ is } < \check{\kappa}\text{-distributive}"$,
- (iii) *Generics over V for P and Q are mutually generic.*

Let $\text{Add}(\alpha, \kappa)$ denote the Cohen forcing adding α Cohen subsets of κ : a condition $p \in \text{Add}(\alpha, \kappa)$ is a function $p : \text{dom}(p) \rightarrow \{0, 1\}$ such that $\text{dom}(p) \subseteq \alpha \times \kappa$, $|\text{dom}(p)| < \kappa$, $p < q$ iff $p \supseteq q$. If κ is a regular cardinal, then $\text{Add}(\alpha, \kappa)$ is κ^+ -c.c. and $< \kappa$ -closed. An $\text{Add}(\alpha, \kappa)$ -generic filter G is best represented by a subset of $\alpha \times \kappa$. If $A \subseteq \alpha$, then $G \cap (A \times \kappa)$ naturally gives an $\text{Add}(\text{o.t.}(A), \kappa)$ -generic filter. Any bijection $f : \alpha \times \kappa \rightarrow \beta \times \kappa$ gives an isomorphism of $\text{Add}(\alpha, \kappa)$ and $\text{Add}(\beta, \kappa)$; if $G \subseteq \alpha \times \kappa$ is $\text{Add}(\alpha, \kappa)$ -generic, then $f[G]$ is $\text{Add}(\beta, \kappa)$ -generic.

Proof of Theorem 1. Let P_κ be an Easton product of $\langle \text{Add}(1, \lambda^+); \lambda \in \text{Reg}(\kappa) \rangle$; we use the trivial forcing if $\lambda < \kappa$ is not a regular cardinal. Since P_κ is a direct limit, it can be considered to be a subset of V_κ . For regular λ factor P_κ as $P_\lambda \times P_{\lambda, \kappa}$; then P_λ is λ^+ -c.c. and $P_{\lambda, \kappa}$ is λ -closed. Consequently by Easton's Lemma

$$P_\lambda \Vdash \text{"}\check{P}_{\lambda, \kappa} \text{ is } \check{\lambda}\text{-distributive"}.$$

It follows that P_κ preserves cardinalities, cofinalities, and GCH. As a matter of fact P_κ is the well-known Kunen-Paris forcing [KuP71]. Let G be P_κ -generic over V .

Assume $P \in V[G]$ is a given well-founded poset of cardinality $\leq \kappa^+$. I claim that P can be embedded into $\triangleleft_\kappa^{V[G]}$.

Claim 1. *Let $U_0 \in V$ be a measure on κ , $j_0 : V \rightarrow \text{Ult}(V, U_0) = M_0$. Then j_0 can be lifted to an elementary embedding $j_0^* : V[G] \rightarrow M_0[G \times \tilde{H}_0]$, $j_0^*(G) = G \times \tilde{H}_0$, uniquely determined by $\tilde{H}_0 \in V[G]$.*

Proof. Factor $j_0(P_\kappa)$ as $P_\kappa \times R$ where $R = (j_0 P_\kappa)_{\kappa, j_0 \kappa}$. Since P_κ is a direct limit, it is enough to find an R -generic filter $\tilde{H}_0 \in V$ over M_0 . By Easton's Lemma $\tilde{H}_0$ is then R -generic over $M_0[G]$. The number of R -antichains in M_0 is $j_0(\kappa^+)$, hence the number of these antichains computed in V is only κ^+ . Moreover R is κ -closed in M_0 , thus in V , enabling us to build up an $\tilde{H}_0 \in V$. $\square$

The embedding j_0^* defines a measure $U_0^* \in V[G]$ extending U_0 ; it is easy to see that j_0^* is the ultraproduct embedding given by $U_0^* : M_0 = \{(j_0 g)(\kappa); g \in {}^\kappa V \cap V\}$, and it is enough to prove that $M_0[G \times \tilde{H}_0] = \{(j_0^* g)(\kappa); g \in {}^\kappa V[G] \cap V[G]\}$. Let $x = i_{G \times \tilde{H}_0}(\dot{x}) \in M_0[G \times \tilde{H}_0]$, find $g \in {}^\kappa V \cap V$ such that $(j_0 g)(\kappa) = \dot{x}$, and define $g^*(\alpha) = i_G(g(\alpha))$ if $g(\alpha)$ as a P_κ -name, then $g^* \in {}^\kappa V[G] \cap V[G]$ and $(j_0^* g^*)(\kappa) = x$.

We consider only one-step extensions of this type. Notice that the forcing R defined above is a product of forcings that always starts with $\text{Add}(1, \kappa^+)^V$. Thus we can always factor $\tilde{H}_0$ as $H_0^\kappa \times H_0$ where H_0^κ is $\text{Add}(1, \kappa^+)$ -generic over $M_0[G]$. Now we need to find some sufficient and necessary conditions for $\triangleleft$ on those extensions.

Claim 2. *Let U_0^*, U_1^* be extensions of U_0, U_1 given by $H_0^\kappa \times H_0$ and $H_1^\kappa \times H_1$. If $U_0^* \triangleleft U_1^*$, then $U_1 \neq U_0, U_1 \not\triangleleft U_0$, and $H_0^\kappa \times H_0 \in M_1[G \times H_1^\kappa]$. On the other hand if $U_0 \triangleleft U_1$ and $H_0^\kappa \times H_0 \in M_1[G \times H_1^\kappa]$, then $U_0^* \triangleleft U_1^*$.*

Proof. Assume $U_0 \triangleleft U_1$ and $H_0^\kappa \times H_0 \in M_1[G \times H_1^\kappa]$. Extend $j_0 \upharpoonright M_1 = j_0' : M_1 \rightarrow M_0^1 = \text{Ult}(M_1, U_0)$ to $\tilde{j}_0 : M_1[G] \rightarrow M_0^1[G \times H_0^\kappa \times H_0]$ in $M_1[G \times H_1^\kappa]$. Then $\tilde{j}_0 = j_0^* \upharpoonright M_1[G]$ defines the measure U_0^* in $M_1[G \times H_1^\kappa]$ because subsets of κ are the same in $V[G]$ and $M_1[G]$.

Assume that $U_0^* \triangleleft U_1^*$; then $j_0(\kappa) < j_1(\kappa)$, and so U_1 cannot be less than U_0 in the Mitchell ordering. $U_0^* \in M_1[G \times H_1^\kappa]$ since the rest of the forcing $(j_0 P_\kappa)_{\kappa+1, j_0 \kappa}$ is sufficiently closed. Consequently $j_0^*(G) = G \times H_0^\kappa \times H_0 \in M_1[G \times H_1^\kappa]$. $\square$

Let $P = (\Theta, <_P)$, $\Theta < \kappa^{++}$, so that the ordering of ordinals extends $<_P$. Using the κ -c.c. of P_κ find a P_κ -name $\dot{P} \subseteq \Theta \times \Theta \times P_\kappa$ for P of cardinality $\leq \kappa^+$. Since the measures on κ cover $\mathcal{P}(\kappa^+)$, there is a $\beta < \kappa^{++}$ so that $\dot{P} \in \text{Ult}(V, U_\beta^\kappa)$ and $\Theta < \kappa^{++ + \text{Ult}(V, U_\beta^\kappa)}$. Fix a sequence of measures $U_0 \triangleleft U_1 \triangleleft \dots \triangleleft U_\alpha \triangleleft \dots$ ($\alpha < o(P)$) starting with $U_0 = U_\beta^\kappa$. Denote $j_\alpha : V \rightarrow \text{Ult}(V, U_\alpha) = M_\alpha$; then $P \in M_\alpha[G]$ and $\Theta < \kappa^{++ + M_\alpha}$ for all $\alpha < o(P)$ by the choice of U_0 and Fact 1. I am going to find $U_p^* \in V[G]$ (for $p \in P$) extending $U_{o_P(p)}$ so that $U_p^* \triangleleft U_q^*$ iff $p <_P q$.

Claim 3. *There is $\tilde{H} \in V[G]$ simultaneously $\text{Add}(1, \kappa^+)$ -generic over all $M_\alpha[G]$ for $\alpha < o(P)$.*

Proof. The number of $\text{Add}(1, \kappa^+)$ -antichains in $M_\alpha[G]$ computed in $V[G]$ is κ^+ for a fixed α , hence still κ^+ for all $\alpha < o(P) < \kappa^{++}$ together. $\text{Add}(1, \kappa^+)$ is κ -closed; thus the generic $\tilde{H}$ can be constructed in $V[G]$. $\square$

Let $\pi : 1 \times \kappa^+ \rightarrow \Theta \times \kappa^+$ be a bijection in M_0 . Then $\pi[\tilde{H}]$ is $\text{Add}(\Theta, \kappa^+)$ -generic over all $M_\alpha[G]$. For $q \in P$ let

$$H_q^\kappa = \pi_q^{-1}[\pi[\tilde{H}] \cap (\{p; p \leq_P q\} \times \kappa^+)]$$

where $\pi_q : 1 \times \kappa^+ \rightarrow \{p; p \leq_P q\} \times \kappa^+$ is a bijection in $M_0[G]$ (and so in all $M_\alpha[G]$). H_q^κ is obviously $\text{Add}(1, \kappa^+)$ -generic over all $M_\alpha[G]$. Moreover for $p \in P$ let g_p denote the p -th $\text{Add}(1, \kappa^+)$ -generic of $\pi[\tilde{H}]$,

$$g_p = \{\alpha < \kappa; (p, \alpha) \in \pi[\tilde{H}]\}.$$

If $p \leq_P q$, then obviously $g_p \in M_\alpha[G \times H_q^\kappa]$ and, on the other hand, if $p \not\leq_P q$, then g_p is $\text{Add}(1, \kappa^+)$ -generic over $M_\alpha[G \times H_q^\kappa]$, and so $g_p \notin M_\alpha[G \times H_q^\kappa]$. To complete the definition of U_q^* we need to find an appropriate $(j_\alpha P_\kappa)_{\kappa+1, j_\alpha \kappa}$ -generic filter over M_α , where $\alpha = o_P(q)$. Consider $j'_\alpha : M_{\alpha+1} \rightarrow \text{Ult}(M_{\alpha+1}, U_\alpha) = M'_\alpha$, and find an $H_q \in M_{\alpha+1}$ which is $(j_\alpha P_\kappa)_{\kappa+1, j_\alpha \kappa}$ -generic over M'_α as in Claim 1. By Fact 1 $V_{j_\alpha(\kappa)+1}^{M_\alpha} = V_{j_\alpha(\kappa)+1}^{M'_\alpha}$, thus H_q is generic over M_α as well. $G \times H_q^\kappa \times H_q$ is $j_\alpha P_\kappa$ -generic over M_α , defining an extension of j_α

$$j_\alpha^* : V[G] \rightarrow M_\alpha[G \times H_q^\kappa \times H_q].$$

The embedding j_α^* defines a measure $U_q^* \in V[G]$ extending U_α ; U_q^* is actually an element of $M_{\alpha+1}[G \times H_q^\kappa]$.

Claim 4. $U_p^* \triangleleft U_q^*$ iff $p <_P q$.

Proof. Let $U_p^* \triangleleft U_q^*$, then by Claim 2 $\alpha = o_P(p)$ must be strictly less than $\beta = o_P(q)$, and $H_p^\kappa \times H_p \in M_\beta[G \times H_q^\kappa]$. Since g_p can be decoded from H_p^κ using $\pi_p \in M_\beta[G]$, it is in $M_\beta[G \times H_q^\kappa]$, which is possible only if $p \leq_P q$, i.e. $p <_P q$.

Let $p <_P q$; then $\alpha = o_P(p) < o_P(q) = \beta$. According to Claim 2 all we need to prove is that $H_p^\kappa \times H_p \in M_\beta[G \times H_q^\kappa]$. The mappings π_p, π_q are both in $M_\beta[G]$. Consequently we can compute

$$H_p^\kappa = \pi_p^{-1}[\pi_q[H_q^\kappa] \cap (\{p'; p' \leq_P p\} \times \kappa^+)]$$

in $M_\beta[G \times H_q^\kappa]$. By the construction H_p is in $M_{\alpha+1}$, and so in M_β by Fact 1. $\square$

Let us prove that there is a Laver's function on κ in $V[G]$.

First find a sequence of bijections $\pi_\lambda : P_\lambda \rightarrow \lambda$ ($\lambda \leq \kappa$ inaccessible) coherent in the following sense:

$$(1) \quad \forall \lambda \leq \kappa \forall \alpha < o^\mathcal{U}(\lambda) : j_{U_\alpha^\lambda}(\langle \pi_{\lambda'}; \lambda' < \lambda \rangle)(\lambda) = \pi_\lambda.$$

Assume $\pi_{\lambda'}$ has been defined for $\lambda' < \lambda$. If $o^\mathcal{U}(\lambda) = 0$ pick any bijection $\pi_\lambda : P_\lambda \rightarrow \lambda$. If $o^\mathcal{U}(\lambda) > 0$, then π_λ can be defined as $j_{U_0^\lambda}(\langle \pi_{\lambda'}; \lambda' < \lambda \rangle)(\lambda)$ since $j_{U_0^\lambda}(\langle \pi_{\lambda'}; \lambda' < \lambda \rangle)(\lambda) = P_\lambda$. Condition (1) is then satisfied because $\vec{U}$ forms a coherent sequence of measures. Notice that if σ_λ is the bijection $\lambda^+ \times \lambda \rightarrow \lambda^+$ given by the maximolexicographical ordering of $\lambda^+ \times \lambda$, then also

$$j_{U_\alpha^\lambda}(\langle \sigma_{\lambda'}; \lambda' < \lambda \rangle)(\lambda) = \sigma_\lambda \quad \text{for any } \alpha < o^\mathcal{U}(\lambda).$$

A P_λ -name $\dot{x} \subseteq \lambda^+ \times P_\lambda$ for a subset of λ^+ can be uniquely coded using π_λ and σ_λ as a subset of λ^+ . If $f(\lambda)$ codes in this way a P_λ -name $\dot{x}$ for a subset of λ^+ define $f^*(\lambda) = i_{G \restriction P_\lambda}(\dot{x})$. It is easy to verify that this defines a Laver's function on κ in $V[G]$. Theorem 1 is proved. $\square$

We can still ask what well-founded κ^{++} -like posets can be embedded into $\triangleleft$. For example, can we embed the poset consisting of a chain of length κ^{++} and one incomparable element? The answer is positive; however, in this case we have to destroy the covering of $\mathcal{P}(\kappa^+)$.

Theorem 2. Assume that $V = L[\vec{U}]$, $\delta^{\mathcal{M}}(\kappa) = \kappa^{++}$. Then there is a generic extension $V[G]$ preserving cardinalities, cofinalities, and GCH such that all well-founded κ^{++} -like posets in V and all well-founded posets of cardinality $\leq \kappa^+$ in $V[G]$ are embeddable into $\triangleleft_{\kappa}^{V[G]}$.

Proof. Let $G = H \times \tilde{G}$ be $\text{Add}(\kappa^{++}, \kappa^+) \times P_{\kappa}$ -generic over V where P_{κ} is the Kunen-Paris forcing. Cardinalities, cofinalities, and GCH are obviously preserved in $V[G]$.

If $U_0 \in V$ is a measure on κ , $j_0 : V \rightarrow \text{Ult}(V, U_0) = M_0$, and $j_0^* : V[\tilde{G}] \rightarrow M_0[\tilde{G} \times H_0^{\kappa} \times H_0]$ an extension of j_0 defined in $V[\tilde{G} \times H]$, then $U_0^* = \{X \in V[\tilde{G}]; X \subseteq \kappa \text{ and } \kappa \in j_0^*(X)\}$ is a measure in $V[\tilde{G} \times H]$ because H does not add any new subsets of κ . Similarly as in the proof of Theorem 1 if U_0^*, U_1^* are two such extensions of U_0, U_1 given by $H_0^{\kappa} \times H_0, H_1^{\kappa} \times H_1$, then $U_0^* \triangleleft U_1^*$ implies $U_1 \neq U_0, U_1 \not\triangleleft U_0, H_0^{\kappa} \times H_0 \in M_1[G \times H_1^{\kappa}]$. On the other hand $U_0 \triangleleft U_1, H_0^{\kappa} \times H_0 \in M_1[G \times H_1^{\kappa}]$ implies $U_1^* \triangleleft U_0^*$.

Let $P \in V$ be a κ^{++} -like poset.

Claim. P can be enumerated as $\langle p_{\alpha}; \alpha < \kappa^{++} \rangle$ so that $p_{\alpha} <_P p_{\beta}$ implies $\alpha < \beta$. Consequently there is a sequence of measures $U_0 \triangleleft \dots \triangleleft U_{\alpha} \triangleleft \dots$ ($\alpha < \kappa^{++}$) on κ so that

$$\forall \alpha < \kappa^{++} : M_{\alpha} \models \text{"}|\alpha| \leq \kappa^+ \text{" and } \{\gamma; p_{\gamma} \leq_P p_{\alpha}\} \in M_{\alpha},$$

where $j_{\alpha} : V \rightarrow M_{\alpha} = \text{Ult}(V, U_{\alpha})$.

Proof. Define the enumeration by induction: Start with any well-ordering $\prec$ of P of order type κ^{++} . Assume that $\langle p_{\alpha}; \alpha < \mu \rangle$ has been defined so that

$$\forall \alpha < \mu : \{q \in P; q <_P p_{\alpha}\} \subseteq \{p_{\gamma}; \gamma < \alpha\}.$$

Let p_{μ} be the $\prec$ -first element of $P \setminus \{p_{\alpha}; \alpha < \mu\}$ that is minimal with respect to $<_P$. Then the sequence $\langle p_{\alpha}; \alpha < \kappa^{++} \rangle$ clearly exhausts all elements of P : Assume that $\{p_{\alpha}; \alpha < \kappa^{++}\} \subsetneq P$; let $p \in P \setminus \{p_{\alpha}; \alpha < \kappa^{++}\}$ be a $<_P$ -minimal element. Find $\mu < \kappa^{++}$ such that

$$\{q \in P; q <_P p\} \cup \{p_{\alpha}; \alpha < \kappa^{++} \& p_{\alpha} \prec p\} \subseteq \{p_{\alpha}; \alpha < \mu\}.$$

Then p is still $<_P$ -minimal in $P \setminus \{p_{\alpha}; \alpha < \mu\}$ and so $p_{\mu} \preceq p$ —a contradiction.

The second part of the claim follows easily from the fact that the measures on κ cover $\mathcal{P}(\kappa^+)$. $\square$

Define $U_{p_{\alpha}}^* \in V[G]$ extending U_{α} as follows: Put

$$H_{\alpha}^{\kappa} = \pi_{\alpha}[H \cap (\{\gamma; p_{\gamma} \leq_P p_{\alpha}\} \times \kappa^+)]$$

where $\pi_{\alpha} : \{\gamma; p_{\gamma} \leq_P p_{\alpha}\} \times \kappa^+ \rightarrow 1 \times \kappa^+$ is a bijection in M_{α} . H_{α}^{κ} is $\text{Add}(1, \kappa^+)$ -generic over $M_{\alpha}[\tilde{G}]$ —the point is that H is now generic over all $M_{\alpha}[\tilde{G}]$ as $\alpha < \kappa^{++}$. Then find an $H_{\alpha} \in M_{\alpha+1}$ that is $(j_{\alpha} P_{\kappa})_{\kappa+1, j_{\alpha} \kappa}$ -generic over $\text{Ult}(M_{\alpha+1}, U_{\alpha})$. That defines a measure $U_{p_{\alpha}}^*$ extending U_{α} . Similarly as in the proof of Theorem 1 $U_{p_{\alpha}}^* \triangleleft U_{p_{\beta}}^*$ iff $p_{\alpha} <_P p_{\beta}$.

Finally let $P \in V[G]$ be a well-founded poset of cardinality κ^+ . Then by the κ^{++} -c.c. of $\text{Add}(\kappa^{++}, \kappa^+)$ there is $\vartheta < \kappa^{++}$ so that $P \in V[\tilde{G} \times (H \restriction \vartheta)]$. Represent

P as $([\vartheta, \Theta], <_P)$, $\vartheta < \Theta < \kappa^{++}$, and find $U_0 \triangleleft \cdots \triangleleft U_\alpha \triangleleft \cdots$ ($\alpha < o(P)$) so that $P \in M_0[\tilde{G} \times (H \restriction \vartheta)]$ and $\vartheta < \kappa^{++M_0}$. Then define U_q^* extending the $U_{o_P(q)}$ setting

$$H_q^\kappa = \pi_q[H \cap ((\vartheta \cup \{p; p <_P q\}) \times \kappa^+)]$$

where $\pi_q : (\vartheta \cup \{p; p <_P q\}) \times \kappa^+ \rightarrow 1 \times \kappa^+$ is a bijection in $M_0[\tilde{G} \times (H \restriction \vartheta)]$ such that $\pi_q \restriction (\vartheta \times \kappa^+)$ is in M_0 . Find H_q as in the proof of Theorem 1. Then similarly as in the proof $H_p^\kappa \in M_\beta[\tilde{G} \times H_q^\kappa]$ iff $p <_P q$ iff $U_p^* \triangleleft U_q^*$. $\square$

Remark 1. J. Cummings in [Cu93] starts with $V = K[\vec{U}_{\max}]$ ($\mathcal{o}^U(\kappa) < \kappa^{++}$), then applies a forcing similar to the Kunen-Paris forcing, and classifies all measures in $V[G]$ using special properties of $K[\vec{U}_{\max}]$. More specifically the measures are divided into blocks of incomparable measures $M(\alpha, \beta)$ ($\alpha < \beta \leq \mathcal{o}^U(\kappa)$) of cardinality κ^+ or κ^{++} so that for $U \in M(\alpha, \beta)$, $W \in M(\gamma, \delta) : U \triangleleft W$ iff $\beta \leq \gamma$.

Starting with $V = K[\vec{U}_{\max}]$ we can use the same method to classify measures in $V[G]$ where G is P_κ -generic over V .

A finite normal iteration $j : V \rightarrow N$ of length $n+1$ is an iteration of ultraproducts by measures on $\kappa = \kappa_0 < \kappa_1 < \cdots < \kappa_n$. Any finite normal iteration $j : V \rightarrow N$ that starts with a measure U gives κ^{++} extensions U^* in $V[G]$ of U such that $j_{U^*} \restriction V = j$, $j_{U^*}(G) = G * H^\kappa * H$. All measures in $V[G]$ are produced in this way. We can give sufficient and necessary conditions for $\triangleleft$ in $V[G]$: Let U_0^*, U_1^* extending U_0, U_1 be given by finite normal iterations $j_0 : V \rightarrow N_0$, $j_1 : V \rightarrow N_1$ and $H_0^\kappa * H_0, H_1^\kappa * H_1$. Then $U_0^* \triangleleft U_1^*$ iff $j_0 \restriction \text{Ult}(V, U_1)$ is an internal iteration in $\text{Ult}(V, U_1)$ and

$$H_0^\kappa * H_0 \in \text{Ult}(V, U_1)[G * H_1^\kappa].$$

However, we can hardly describe the Mitchell ordering $\triangleleft$ in $V[G]$ in a simpler manner.

That is illustrated by the following: Let U_0 be the minimal measure on κ in V , and let U_0^* be its one-step extension using $H_0^\kappa * H_0 \in V[G]$. We have seen that there may be measures above U_0^* even if $H_0^\kappa * H_0 \notin M_\alpha[G]$ for all $\alpha < o(\kappa)$ ($M_\alpha = \text{Ult}(V, U_\alpha^\kappa)$). However it is also possible that there are no measures above U_0^* . It follows from the following joint lemma with J. Zapletal.

Lemma. *There is $H_0^\kappa \in V[G]$ $\text{Add}(1, \kappa^+)$ -generic over $M_0[G]$ such that for any $\alpha < \mathcal{o}^U(\kappa)$, $\alpha > 0$, there is no $H_1^\kappa \in V[G]$ $\text{Add}(1, \kappa^+)$ -generic over $M_\alpha[G]$, $\alpha < o(\kappa)$, satisfying $H_0^\kappa \in M_\alpha[G][H_1^\kappa]$.*

Proof. Let $R \subseteq \kappa^+ \times \kappa^+$ be a well-ordering of order type γ where $\gamma > \kappa^{++M_\alpha}$ for all $\alpha < o(\kappa) < \kappa^{++}$. Notice that if H_1^κ is any $\text{Add}(1, \kappa^+)$ -generic over $M_\alpha[G]$, then $R \notin M_\alpha[G][H_1^\kappa]$; otherwise γ would be less than κ^{++M_α} . So it would be enough to code R into H_0^κ . Let $\langle a_\alpha; \alpha < \kappa^+ \rangle \in M_1[G]$ canonically enumerate $\kappa^+ \times \kappa^+$ and $\langle D_\alpha; \alpha < \kappa^+ \rangle \in M_1[G]$ enumerate all $M_0[G]$ -dense subsets of $\text{Add}(1, \kappa^+)$, each set D_α enumerated by ordinals $< \kappa^+$. Construct a descending sequence of conditions $\langle p_\alpha; \alpha < \kappa^+ \rangle \in V[G]$ as follows: Assume $\langle p_\delta; \delta < \alpha \rangle$ has been constructed, then find the first $q \in D_\alpha$ extending $\bigcup \{p_\delta; \delta < \alpha\}$, let $\eta = \sup\{\xi + 1; \xi \in \text{dom}(q)\}$, and put $p \restriction \eta = q$ and

$$p(\eta) = \begin{cases} 1 & \text{iff } a_\alpha \in R, \\ 0 & \text{otherwise.} \end{cases}$$

That gives an $\text{Add}(1, \kappa^+)$ -generic filter H_0^κ over $M_0[G]$ such that for $\alpha > 0$

$$H_0^\kappa \in M_\alpha[G][H_1^\kappa]$$

implies $R \in M_\alpha[G][H_1^\kappa]$. $\square$

Remark 2. It is not true in general that $U_0^* \triangleleft U_1^*$ implies $U_0 \triangleleft U_1$ as Claim 2 in the proof of Theorem 1 might suggest: There is a model M , two measures $U_0, U_1 \in M$ and their extensions U_0^*, U_1^* in $M[\tilde{G}]$, where $\tilde{G}$ is P_κ -generic over M , so that

$$U_0 \not\triangleleft U_1 \text{ but } U_0^* \triangleleft U_1^*.$$

Proof. Start with two measures $U_0 \triangleleft U_1$ in V , and with the corresponding canonical embeddings $j_0 : V \rightarrow M_0$, $j_1 : V \rightarrow M_1$. Let $G \times \tilde{G}$ be $P_\kappa \times P_\kappa$ -generic over V . Then find $H_0^\kappa \times H_1^\kappa \times H_2^\kappa \in V$ a filter $\text{Add}(3, \kappa^+)$ -generic over M_1 , $H_1 \times \tilde{H}_1 \in V$ a filter $(j_1 P_\kappa)_{\kappa+1, j_1 \kappa} \times (j_1 P_\kappa)_{\kappa+1, j_1 \kappa}$ -generic over M_1 , and $H_0 \times \tilde{H}_0 \in M_1$ a filter $(j_0 P_\kappa)_{\kappa+1, j_0 \kappa} \times (j_0 P_\kappa)_{\kappa+1, j_0 \kappa}$ -generic over M_0 . Using Easton's Lemma $G, \tilde{G}, H_0^\kappa, H_1^\kappa, H_2^\kappa, H_1, \tilde{H}_1$ are mutually generic over M_1 and $G, \tilde{G}, H_0^\kappa, H_1^\kappa, H_2^\kappa, H_0, \tilde{H}_0$ are mutually generic over M_0 . First extend U_0, U_1 to U_0^*, U_1^* in $M = V[G]$ so that $j_1^*(G) = G \times H_1^\kappa \times H_1$ and $j_0^*(G) = G \times (H_0^\kappa \otimes H_1^\kappa) \times H_0$ where $H_0^\kappa \otimes H_1^\kappa$ denotes a coding of H_0^κ, H_1^κ into an $\text{Add}(1, \kappa^+)$ -generic. Obviously $U_0^* \not\triangleleft U_1^*$ since $H_0^\kappa \notin M_1[j_1^*(G)]$. Then extend U_0^*, U_1^* to U_0^{**}, U_1^{**} in $M[\tilde{G}] = V[G \times \tilde{G}]$ so that $j_1^{**}(\tilde{G}) = \tilde{G} \times (H_0^\kappa \otimes H_2^\kappa) \times \tilde{H}_1$ and $j_0^{**}(\tilde{G}) = \tilde{G} \times H_2^\kappa \times \tilde{H}_0$. Then $U_0^{**} \triangleleft U_1^{**}$ since $U_0^*, j_0^{**}(\tilde{G}) \in M_1[j_1^{**}(G \times \tilde{G})]$. $\square$

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