A PROPORTIONAL DVORETZKY-ROGERS FACTORIZATION RESULT

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ABSTRACT. If X is an n -dimensional normed space and $\varepsilon \in (0, 1)$, there exists $m \geq (1 - \varepsilon)n$, such that the formal identity $i_{2,\infty}: l_2^m \rightarrow l_\infty^m$ can be written as $i_{2,\infty} = \alpha \circ \beta$, $\beta: l_2^m \rightarrow X$, $\alpha: X \rightarrow l_\infty^m$, with $\|\alpha\| \cdot \|\beta\| \leq c/\varepsilon$. This is proved as a consequence of a Sauer-Shelah type theorem for ellipsoids.

1. INTRODUCTION

A version of the classical Dvoretzky-Rogers lemma [D-R] asserts that, if $(X, \|\cdot\|)$ is an n -dimensional normed space, there exist vectors $x_1, \dots, x_m \in X$, $m = \lfloor \sqrt{n} \rfloor$, such that for any choice of real numbers $t_1, \dots, t_m$,

$$\max_{j \leq m} |t_j| \leq \left\| \sum_{j \leq m} t_j x_j \right\|_X \leq c \left(\sum_{j \leq m} t_j^2 \right)^{1/2},$$

where $c > 0$ is an absolute constant. Towards a strengthening of this result for m proportional to n , Bourgain-Szarek [B-S] and later Szarek-Talagrand [S-T] proved the following:

Theorem 1. *If $(X, \|\cdot\|)$ is an n -dimensional normed space and $\varepsilon \in (0, 1)$, there exist vectors $x_1, \dots, x_m \in X$, $m \geq (1 - \varepsilon)n$, such that for any reals $t_1, \dots, t_m$,*

$$\max_{j \leq m} |t_j| \leq \left\| \sum_{j \leq m} t_j x_j \right\|_X \leq c\varepsilon^{-d} \left(\sum_{j \leq m} t_j^2 \right)^{1/2},$$

where $c, d > 0$ are absolute constants. Equivalently, the formal identity $i_{2,\infty}: l_2^m \rightarrow l_\infty^m$ can be written as $i_{2,\infty} = \alpha \circ \beta$, where $\beta: l_2^m \rightarrow X$, $\alpha: X \rightarrow l_\infty^m$, and $\|\alpha\| \cdot \|\beta\| \leq c\varepsilon^{-d}$. The same holds true for $i_{1,2}: l_1^m \rightarrow l_2^m$.

The best possible dependence on ε is not known. As shown by S. J. Szarek [Sz.1], there exists an n -dimensional normed space X such that $\|\alpha\| \cdot \|\beta\| \geq c(n/\log n)^{1/10}$ whenever $i_{2,\infty}: l_2^n \rightarrow l_\infty^n$ is written as $i_{2,\infty} = \alpha \circ \beta$ (α, β as above), and this implies that d in Theorem 1 has to be at least $1/10$. On the other hand, in [S-T] it is proved that Theorem 1 holds with $d = 2$, and in [G] we obtain a similar result with $d = 3/2$. Here, we shall show that the same holds true with $d = 1$.

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Let us note that the method establishing this “proportional Dvoretzky-Rogers factorization” is closely related to the problem of the Banach-Mazur distance to the cube. A detailed exposition of the techniques used so far for both problems is given in [Sz.2].

The source of the improvement on the estimates in Theorem 1 is a Sauer-Shelah type theorem for ellipsoids, which we feel is of independent interest: The well-known combinatorial Sauer-Shelah lemma [Sa], [Sh] states that if $0 \leq l < s$ and M is a subset of $\{-1, 1\}^s$ of cardinality $|M| > \binom{s}{0} + \binom{s}{1} + \cdots + \binom{s}{l}$, then there exists $\sigma \subset \{1, \dots, s\}$, $|\sigma| > l$, such that $P_\sigma(M) = \{-1, 1\}^\sigma$, where P_σ is the restriction map $(\delta_j)_{j \leq s} \rightarrow (\delta_j)_{j \in \sigma}$. A special case of this lemma is of particular interest: If $M \subset \{-1, 1\}^s$ and $|M| \geq 2^{s-1}$, then we can find $\sigma \subset \{1, \dots, s\}$, $|\sigma| \geq \frac{s}{2}$, with $P_\sigma(M) = \{-1, 1\}^\sigma$.

In connection with their work on the Banach-Mazur distance to the cube, Szarek and Talagrand [S-T] proved an isomorphic variant of the Sauer-Shelah lemma: If $M \subset \{-1, 1\}^s$, viewed now as a set of points in $\mathbf{R}^s$, and if $|M| \geq 2^{s-1}$, $\varepsilon \in (0, 1)$, then there exists $\sigma \subset \{1, \dots, s\}$, $|\sigma| \geq (1 - \varepsilon)s$, such that

$$\text{absconv}(P_\sigma(M)) \supseteq c\varepsilon[-1, 1]^\sigma,$$

where $c > 0$ is an absolute constant (and the absolute convex hull is taken in $\mathbf{R}^\sigma$).

For our purposes, we need to consider the following situation: Let $u_1, \dots, u_s$ be vectors in $\mathbf{R}^n$, of Euclidean norm $|u_j|_n \leq 1$, $j = 1, \dots, s$. Define the symmetric convex set

$$\mathcal{E} = \left\{ (\delta_j)_{j \leq s} \in \mathbf{R}^s : \left| \sum_{j \leq s} \delta_j u_j \right|_n \leq 1 \right\}.$$

(Note that if $s \leq n$ and the vectors u_j are linearly independent in $\mathbf{R}^n$, then $\mathcal{E}$ is an ellipsoid in $\mathbf{R}^s$. This will be the context in the proof of Theorem 1.) Again, we are interested in the “size” of the image $P_\sigma(\mathcal{E})$ of $\mathcal{E}$ for “large” subsets σ of $\{1, \dots, s\}$. Our main result is then the following

Theorem 2. *If $u_j \in \mathbf{R}^n$, $|u_j|_n \leq 1$, $j = 1, \dots, s$, and*

$$\mathcal{E} = \left\{ (\delta_j)_{j \leq s} \in \mathbf{R}^s : \left| \sum_{j \leq s} \delta_j u_j \right|_n \leq 1 \right\},$$

then for every $\varepsilon \in (0, 1)$, we can find $\sigma \subseteq \{1, \dots, s\}$, $|\sigma| \geq (1 - \varepsilon)s$, such that

$$P_\sigma(\mathcal{E}) \supseteq c\sqrt{\varepsilon}D_\sigma,$$

where D_σ is the Euclidean unit ball in $\mathbf{R}^\sigma$ and $c > 0$ is an absolute constant.

We shall use the standard notation from [M-Sc] or [T-J]. By $|\cdot|$ we denote the cardinality of a finite set. The letter c will always denote an absolute positive constant, not necessarily the same in all its occurrences. For basic facts about p -absolutely summing operators, used in the proof of Theorem 2, we refer the reader to [L-T], [Pi], and [T-J].

2. PROOF OF THEOREM 2

First, we introduce some additional notation: The set $S = \{1, \dots, s\}$, as well as $\mathbf{R}^S$, will be fixed throughout the proof. If $\varphi \subseteq S$, then $\mathbf{R}^\varphi = \{(\delta_j)_{j \leq s} \in \mathbf{R}^S : \delta_j = 0 \text{ if } j \notin \varphi\}$. A point in $\mathbf{R}^S$ denoted by $(\delta_j)_{j \in \varphi}$ is assumed to satisfy $\delta_j = 0$ if $j \notin \varphi$.

If τ, φ are disjoint subsets of S and $A \subseteq \mathbf{R}^\varphi$, we sometimes write $\mathbf{0}_\tau \times A$ instead of A to indicate that A is to be understood as a subset of $\mathbf{R}^{\varphi \cup \tau}$. In particular, if $\varphi \subseteq S_1 \subseteq S$ and $L > 0$, then

$$\begin{aligned} I_{L, \varphi, S_1} &= \mathbf{0}_\varphi \times \{-L, L\}^{S_1 \setminus \varphi} \\ &= \left\{ (\delta_j)_{j \in S_1} \in \mathbf{R}^{S_1} : \delta_j = \begin{cases} 0 & \text{if } j \in \varphi, \\ \pm L & \text{if } j \in S_1 \setminus \varphi \end{cases} \right\}. \end{aligned}$$

Note that $|I_{L, \varphi, S_1}| = 2^{|S_1 \setminus \varphi|}$. If $a \in \mathbf{R}^\varphi$, $b \in \mathbf{R}^\tau$, and τ, φ are disjoint subsets of S , then $(a, b) \in \mathbf{R}^{\varphi \cup \tau}$ is the sum $a + b$. Finally, if S_1 is a non-empty subset of S , we define

$$\mathcal{E}_{S_1} = \left\{ (\delta_j)_{j \in S_1} \in \mathbf{R}^{S_1} : \left| \sum_{j \in S_1} \delta_j u_j \right|_n \leq 1 \right\}.$$

Our starting point is then an immediate consequence of the Sauer-Shelah lemma:

Lemma 1. *If $L > 0$, $\varphi \subseteq S_1 \subseteq S$, and $M \subseteq \mathbf{0}_\varphi \times \{-L, L\}^{S_1 \setminus \varphi}$, with $|M| \geq 2^{|S_1 \setminus \varphi| - 1}$, then there exists $\sigma \subseteq S_1 \setminus \varphi$, $|\sigma| \geq \frac{|S_1 \setminus \varphi|}{2}$, such that*

$$P_{\varphi \cup \sigma}(M) = \mathbf{0}_\varphi \times \{-L, L\}^\sigma. \quad \square$$

Using an inductive argument based on Lemma 1, we obtain a first result on the size of the projections of $\mathcal{E}_{S_1}$, for an arbitrary $S_1 \subseteq S$. This step is crucial for our proof of Theorem 2, so we state it as our next lemma and give its proof, although it can essentially be found in [G].

Lemma 2. *If $\emptyset \neq S_1 \subseteq S$ and $\varepsilon \in (0, 1)$ are given, then there exists $\sigma \subseteq S_1$, with $|\sigma| \geq (1 - \varepsilon)|S_1|$, such that*

$$P_\sigma(\mathcal{E}_{S_1}) \supseteq \frac{c\sqrt{\varepsilon}}{\sqrt{|S_1|}}[-1, 1]^\sigma,$$

where $c > 0$ is an absolute constant.

Proof. Set $\alpha_k = \sum_{r=0}^{k-1} 2^{r/2}$, $\beta_k = \sum_{r=0}^{k-1} 2^r = 2^k - 1$, and $Q_\tau = [-1, 1]^\tau$ for every non-empty $\tau \subseteq S_1$.

We shall prove by induction that:

$$\begin{aligned} &\text{For } k = 1, 2, \dots, \text{ one can find } \sigma_k \subseteq S_1, \text{ with } |\sigma_k| \geq \\ (*) \quad &(1 - \frac{1}{2^k})|S_1|, \text{ such that} \\ &Q_{\sigma_k} \subseteq P_{\sigma_k}(\alpha_k \sqrt{2|S_1|} \mathcal{E}_{S_1} \cap \beta_k Q_{S_1}). \end{aligned}$$

Since $\alpha_k \leq \frac{2^{k/2}}{\sqrt{2}-1}$, condition (*) clearly implies that, for $k = 1, 2, \dots$,

$$P_{\sigma_k}(\mathcal{E}_{S_1}) \supseteq \frac{c}{\sqrt{|S_1|}} \sqrt{\frac{1}{2^k}} [-1, 1]^{\sigma_k}$$

with $c = 1 - \frac{1}{\sqrt{2}}$, which is the assertion of the lemma for $\varepsilon = 1/2^k$. The continuous version will easily follow with a worse constant c .

Inductive step. Consider the set $J_k = \mathbf{0}_{\sigma_k} \times \{-2^{k/2}, 2^{k/2}\}^{S_1 \setminus \sigma_k}$, where σ_k is the subset of S_1 given by (*). Note that $|J_k| = 2^{|S_1 \setminus \sigma_k|}$. By the parallelogram law and

the fact that $|S_1 \setminus \sigma_k| \leq |S_1|/2^k$, we have

$$\text{Ave}_{(\delta_j) \in J_k} \left| \sum_{j \in S_1} \delta_j u_j \right|_n^2 = 2^k \sum_{j \in S_1 \setminus \sigma_k} |u_j|_n^2 \leq |S_1|,$$

and Markov's inequality implies that there exists $M^{k+1} \subseteq J_k \cap \sqrt{2|S_1|}\mathcal{E}_{S_1}$ with $|M^{k+1}| \geq 2^{|S_1 \setminus \sigma_k| - 1}$. Then, by Lemma 1, we can find $\sigma_{k+1}^* \subseteq S_1 \setminus \sigma_k$, of cardinality $|\sigma_{k+1}^*| \geq \frac{|S_1 \setminus \sigma_k|}{2}$, for which

$$P_{\sigma_k \cup \sigma_{k+1}^*}(M^{k+1}) = \mathbf{0}_{\sigma_k} \times \{-2^{k/2}, 2^{k/2}\}^{\sigma_{k+1}^*}.$$

Since $M^{k+1} \subseteq \sqrt{2|S_1|}\mathcal{E}_{S_1} \cap 2^{k/2}Q_{S_1}$, it follows that

$$(**) \quad \mathbf{0}_{\sigma_k} \times 2^k Q_{\sigma_{k+1}^*} \subseteq P_{\sigma_k \cup \sigma_{k+1}^*}(2^{k/2} \sqrt{2|S_1|}\mathcal{E}_{S_1} \cap 2^k Q_{S_1}).$$

Suppose now that $a \in Q_{\sigma_k}$, $b \in Q_{\sigma_{k+1}^*}$. From the inductive hypothesis (*), there exists $t_a \in \alpha_k \sqrt{2|S_1|}\mathcal{E}_{S_1} \cap \beta_k Q_{S_1}$ such that $P_{\sigma_k}(t_a) = a$. Let $w_a = P_{\sigma_{k+1}^*}(t_a)$; then $w_a \in \beta_k Q_{\sigma_{k+1}^*}$ and

$$(a, w_a) \in P_{\sigma_k \cup \sigma_{k+1}^*}(\alpha_k \sqrt{2|S_1|}\mathcal{E}_{S_1} \cap \beta_k Q_{S_1}).$$

If $v_{a,b} = b - w_a$, it is clear that $v_{a,b} \in Q_{\sigma_{k+1}^*} + \beta_k Q_{\sigma_{k+1}^*} = 2^k Q_{\sigma_{k+1}^*}$, and therefore, by (**),

$$(\mathbf{0}_{\sigma_k}, v_{a,b}) \in P_{\sigma_k \cup \sigma_{k+1}^*}(2^{k/2} \sqrt{2|S_1|}\mathcal{E}_{S_1} \cap 2^k Q_{S_1}).$$

Then,

$$\begin{aligned} (a, b) &= (a, w_a) + (\mathbf{0}_{\sigma_k}, v_{a,b}) \in P_{\sigma_k \cup \sigma_{k+1}^*}(\alpha_k \sqrt{2|S_1|}\mathcal{E}_{S_1} \cap \beta_k Q_{S_1}) \\ &\quad + P_{\sigma_k \cup \sigma_{k+1}^*}(2^{k/2} \sqrt{2|S_1|}\mathcal{E}_{S_1} \cap 2^k Q_{S_1}) \\ &\subseteq P_{\sigma_k \cup \sigma_{k+1}^*}(\alpha_{k+1} \sqrt{2|S_1|}\mathcal{E}_{S_1} \cap \beta_{k+1} Q_{S_1}). \end{aligned}$$

Since $a \in Q_{\sigma_k}$, $b \in Q_{\sigma_{k+1}^*}$ were arbitrary, this means that

$$Q_{\sigma_k \cup \sigma_{k+1}^*} \subseteq P_{\sigma_k \cup \sigma_{k+1}^*}(\alpha_{k+1} \sqrt{2|S_1|}\mathcal{E}_{S_1} \cap \beta_{k+1} Q_{S_1}).$$

If we define $\sigma_{k+1} = \sigma_k \cup \sigma_{k+1}^*$, we readily see that $|\sigma_{k+1}| \geq (1 - \frac{1}{2^{k+1}})|S_1|$, and this completes the inductive step. The first step ($k = 1$) is much simpler. $\square$

For our next two lemmas we shall need to assume that the vectors $u_1, \dots, u_s$ are linearly independent.

Lemma 3. *Let S_1 be a non-empty subset of S . Then, for every $\theta \in (0, \frac{1}{4})$, we can find disjoint $\sigma, \tau \subseteq S_1$ with $|\sigma| \geq \frac{|S_1|}{2}$, $|\tau| \leq \theta|S_1|$, and*

$$P_{S_1 \setminus \tau}(\mathcal{E}_{S_1}) \supseteq \mathbf{0}_{S_1 \setminus (\sigma \cup \tau)} \times c\sqrt{\theta}D_\sigma,$$

where $c > 0$ is an absolute constant.

Proof. Set $V_{S_1} = \text{span}\{u_j, j \in S_1\}$. Then, there exist $x_i \in V_{S_1}, i \in S_1$, such that

$$\langle x_i, u_j \rangle = \delta_{ij} \quad \text{for any pair of } i, j \in S_1.$$

Applying Lemma 2 for the ellipsoid $\mathcal{E}_{S_1}$, we obtain $\tau \subseteq S_1, |\tau| \leq \theta|S_1|$, for which

$$P_{S_1 \setminus \tau}(\mathcal{E}_{S_1}) \supseteq \frac{c\sqrt{\theta}}{\sqrt{|S_1|}}[-1, 1]^{S_1 \setminus \tau}.$$

Then, for any choice of scalars $(t_i)_{i \in S_1 \setminus \tau}$, we can find a vector $(\delta_j)_{j \in S_1}$ in $\mathcal{E}_{S_1}$ whose restriction in $\mathbf{R}^{S_1 \setminus \tau}$ is $(\frac{c\sqrt{\theta}}{\sqrt{|S_1|}} \text{sign } t_j)_{j \in S_1 \setminus \tau}$. In view of the orthogonality relations between the x_i 's and the u_j 's we see that

$$\begin{aligned} \sum_{i \in S_1 \setminus \tau} |t_i| &= \left\langle \sum_{i \in S_1 \setminus \tau} t_i x_i, \sum_{j \in S_1 \setminus \tau} (\text{sign } t_j) u_j \right\rangle \\ &= \frac{\sqrt{|S_1|}}{c\sqrt{\theta}} \left\langle \sum_{i \in S_1 \setminus \tau} t_i x_i, \sum_{j \in S_1} \delta_j u_j \right\rangle \\ &\leq \frac{\sqrt{|S_1|}}{c\sqrt{\theta}} \left\| \sum_{i \in S_1 \setminus \tau} t_i x_i \right\|_n \left\| \sum_{j \in S_1} \delta_j u_j \right\|_n \\ &\leq \frac{\sqrt{|S_1|}}{c\sqrt{\theta}} \left\| \sum_{i \in S_1 \setminus \tau} t_i x_i \right\|_n. \end{aligned}$$

It follows that the operator $T: \text{span}\{x_i, i \in S_1 \setminus \tau\} \subset l_2^n \rightarrow l_1^{|S_1 \setminus \tau|}$, defined by $Tx_i = e_i$ (where $\{e_i\}$ is the canonical orthonormal basis in $\mathbf{R}^{|S_1 \setminus \tau|}$), has norm not exceeding $\sqrt{|S_1|}/c\sqrt{\theta}$. Then, $T^*: l_\infty^{|S_1 \setminus \tau|} \rightarrow l_2^n$ is a 2-absolutely summing operator with 2-summing norm $\pi_2(T^*) \leq K_G \frac{\sqrt{|S_1|}}{c\sqrt{\theta}}$, where K_G is Grothendieck's constant. From Pietch's factorization theorem, applied in the same context as in the proof of Theorem 1.2 [B-T], we can find positive real numbers $\lambda_i, i \in S_1 \setminus \tau$, with $\sum_{i \in S_1 \setminus \tau} \lambda_i^2 = 1$, such that, for any reals $t_i, i \in S_1 \setminus \tau$,

$$\left(\sum_{i \in S_1 \setminus \tau} \left(\frac{t_i}{\lambda_i} \right)^2 \right)^{1/2} \leq K_G \frac{\sqrt{|S_1|}}{c\sqrt{\theta}} \left\| \sum_{i \in S_1 \setminus \tau} t_i x_i \right\|_n.$$

Since $\sum_{i \in S_1 \setminus \tau} \lambda_i^2 = 1$ and $\theta < \frac{1}{4}$, we apply Markov's inequality to obtain $\sigma \subseteq S_1 \setminus \tau, |\sigma| \geq \frac{|S_1|}{2}$, with $\lambda_i \leq \frac{2}{\sqrt{|S_1|}}$ for every $i \in \sigma$. Suppose now that $(\delta_j)_{j \in \sigma} \in D_\sigma$ is given, i.e. $\sum_{j \in \sigma} \delta_j^2 \leq 1$. The set $\{u_j, j \in \tau\} \cup \{x_i, i \in S_1 \setminus \tau\}$ is linearly independent (hence a basis) in V_{S_1} , so we can write

$$\sum_{j \in \sigma} \delta_j u_j + \sum_{j \in \tau} \rho_j u_j = \sum_{i \in S_1 \setminus \tau} t_i x_i$$

for suitable $(\rho_j)_{j \in \tau}, (t_i)_{i \in S_1 \setminus \tau}$. Then,

$$\begin{aligned}
\left| \sum_{j \in \sigma} \delta_j u_j + \sum_{j \in \tau} \rho_j u_j \right|_n^2 &= \left\langle \sum_{j \in \sigma} \delta_j u_j + \sum_{j \in \tau} \rho_j u_j, \sum_{i \in S_1 \setminus \tau} t_i x_i \right\rangle \\
&= \left\langle \sum_{j \in \sigma} \delta_j u_j, \sum_{i \in \sigma} t_i x_i \right\rangle = \sum_{i \in \sigma} \delta_i t_i \leq \left(\sum_{i \in \sigma} t_i^2 \right)^{1/2} \\
&\leq \left(\sum_{i \in \sigma} \left(\frac{t_i}{\lambda_i} \right)^2 \right)^{1/2} \frac{2}{\sqrt{|S_1|}} \leq \left(\sum_{i \in S_1 \setminus \tau} \left(\frac{t_i}{\lambda_i} \right)^2 \right)^{1/2} \frac{2}{\sqrt{|S_1|}} \\
&\leq \frac{2K_G}{c} \frac{1}{\sqrt{\theta}} \left| \sum_{i \in S_1 \setminus \tau} t_i x_i \right|_n,
\end{aligned}$$

and therefore,

$$\left| \sum_{j \in \sigma} \delta_j u_j + \sum_{j \in \tau} \rho_j u_j \right|_n \leq \frac{2K_G}{c} \frac{1}{\sqrt{\theta}}.$$

This means that $\mathbf{0}_{S_1 \setminus (\sigma \cup \tau)} \times (\delta_j)_{j \in \sigma} \in \frac{1}{c_1 \sqrt{\theta}} P_{S_1 \setminus \tau}(\mathcal{E}_{S_1})$ with $c_1 = c/2K_G$. Since $(\delta_j)_{j \in \sigma}$ was arbitrary in D_σ , the lemma follows. $\square$

We are now ready to prove Theorem 2 in the case of independent u_j 's:

Lemma 4. *For every $\varepsilon \in (0, 1)$ one can find $\sigma \subseteq S, |\sigma| \geq (1 - \varepsilon)s$, such that*

$$P_\sigma(\mathcal{E}) \supseteq c\sqrt{\varepsilon}D_\sigma,$$

where $c > 0$ is an absolute constant.

Proof. Given $\varepsilon \in (0, 1)$, we set $\theta = \varepsilon/7$. Let also k be the non-negative integer for which $\frac{1}{2^{k+1}} \leq \varepsilon < \frac{1}{2^k}$. To obtain σ , we shall follow an inductive procedure based on Lemma 3:

Step 1: We set $S_0 = S$, and $\theta_1 = \theta$. Since $\theta_1 \in (0, \frac{1}{4})$, we can find a pair (σ_1, τ_1) of disjoint subsets of S_0 , with $|\tau_1| \leq \theta_1 |S_0|, |\sigma_1| \geq \frac{1}{2} |S_0|$, and $P_{S_0 \setminus \tau_1}(\mathcal{E}_{S_0}) \supseteq \mathbf{0}_{S_0 \setminus (\sigma_1 \cup \tau_1)} \times c\sqrt{\theta_1} D_{\sigma_1}$, where c is the constant from Lemma 3. Finally, we define $S_1 = S_0 \setminus (\sigma_1 \cup \tau_1)$. Note that $|S_1| \leq \frac{1}{2} |S_0| = \frac{s}{2}$.

Inductive step: Suppose that S_l has been defined, and $|S_l| > \frac{\varepsilon}{2}s$. If, in addition, $l < k + 2$, we define $\theta_{l+1} = 2^{l/2}\theta$. Note that then $\theta_{l+1} \leq \frac{\sqrt{2}}{7} 2^{k/2} \varepsilon < \frac{\sqrt{2}}{7} \sqrt{\varepsilon} < \frac{1}{4}$, and therefore we can apply Lemma 3 for $\mathcal{E}_{S_l}$ and θ_{l+1} to obtain a pair $(\sigma_{l+1}, \tau_{l+1})$ of disjoint subsets of S_l , with $|\tau_{l+1}| \leq \theta_{l+1} |S_l|, |\sigma_{l+1}| \geq \frac{1}{2} |S_l|$, and $P_{S_l \setminus \tau_{l+1}}(\mathcal{E}_{S_l}) \supseteq \mathbf{0}_{S_l \setminus (\sigma_{l+1} \cup \tau_{l+1})} \times c\sqrt{\theta_{l+1}} D_{\sigma_{l+1}}$. To complete the inductive step, we define $S_{l+1} = S_l \setminus (\sigma_{l+1} \cup \tau_{l+1})$. Note also that $|S_{l+1}| \leq \frac{1}{2} |S_l|$, hence, as far as we continue performing these steps, $|S_l| \leq \frac{s}{2^l}$.

We end this inductive construction when we arrive at a set S_l of cardinality $|S_l| \leq \frac{\varepsilon}{2}s$. This will certainly happen after at most $(k + 2)$ -steps, since $\frac{1}{2^{k+2}} \leq \frac{\varepsilon}{2}$ and our construction implies that $|S_l| < \frac{s}{2^l}$ for every admissible l .

Suppose l_* is the first index for which $|S_{l_*}| \leq \frac{\varepsilon}{2}s$. We define $\sigma = \sigma_1 \cup \dots \cup \sigma_{l_*}$.

Claim 1. $|\sigma| \geq (1 - \varepsilon)s$.

[*Proof.* Note that $\bigcup_{1 \leq l \leq l_*} (\sigma_l \cup \tau_l) = S \setminus S_{l_*}$, hence

$$\begin{aligned} |S \setminus \sigma| &= |S_{l_*}| + \sum_{1 \leq l \leq l_*} |\tau_l| \leq \frac{\varepsilon}{2}s + \sum_{1 \leq l \leq l_*} \theta_l |S_{l-1}| \\ &\leq \frac{\varepsilon}{2}s + \theta \sum_{1 \leq l \leq l_*} 2^{(l-1)/2} \frac{s}{2^{l-1}} < \frac{\varepsilon}{2}s + \frac{\varepsilon}{7}s \left(\sum_{l=0}^{\infty} \frac{1}{2^{l/2}} \right) < \varepsilon s. \end{aligned}$$

Claim 2. If $1 \leq l \leq l_*$, then

$$P_\sigma(\mathcal{E}) \supseteq \mathbf{0}_{\sigma \setminus \sigma_l} \times c 2^{(l-1)/4} \sqrt{\theta} D_{\sigma_l}.$$

[*Proof.* Suppose that $\Delta_l = (\delta_j)_{j \in \sigma_l} \in c 2^{(l-1)/4} \sqrt{\theta} D_{\sigma_l}$. Then our construction implies that $\mathbf{0}_{S_{l-1} \setminus (\sigma_l \cup \tau_l)} \times \Delta_l \in P_{S_{l-1} \setminus \tau_l}(\mathcal{E}_{S_{l-1}})$. Hence, we can find $(\zeta_i)_{i \in \tau_l}$ such that

$$\left| \sum_{j \in \sigma_l} \delta_j u_j + \sum_{i \in \tau_l} \zeta_i u_i \right|_n \leq 1.$$

Since $\sigma \cap \tau_l = \emptyset$, it is clear that $\mathbf{0}_{\sigma \setminus \sigma_l} \times \Delta_l \in P_\sigma(\mathcal{E})$.

To conclude the proof of the lemma, suppose that $\Delta = (\delta_j)_{j \in \sigma}$ is an arbitrary point in D_σ , i.e. $\sum_{j \in \sigma} \delta_j^2 \leq 1$. Consider the restriction $\Delta_l = \mathbf{0}_{\sigma \setminus \sigma_l} \times (\delta_j)_{j \in \sigma_l}$ of Δ in $\mathbf{R}^{\sigma_l}$, and set $|\Delta_l| = (\sum_{j \in \sigma_l} \delta_j^2)^{1/2}$, $1 \leq l < l_*$. By Claim 2, each Δ_l belongs to $\frac{|\Delta_l|}{c 2^{(l-1)/4} \sqrt{\theta}} P_\sigma(\mathcal{E})$; thus

$$\begin{aligned} \Delta &= \sum_{1 \leq l \leq l_*} \Delta_l \in \left(\sum_{1 \leq l \leq l_*} \frac{|\Delta_l|}{c 2^{(l-1)/4} \sqrt{\theta}} \right) P_\sigma(\mathcal{E}) \\ &\subseteq \frac{1}{c \sqrt{\theta}} \left(\sum_{1 \leq l \leq l_*} |\Delta_l|^2 \right)^{1/2} \left(\sum_{l=0}^{\infty} \frac{1}{2^{l/2}} \right)^{1/2} P_\sigma(\mathcal{E}) \\ &\subseteq \frac{\sqrt{7}}{c} \left(\frac{\sqrt{2}}{\sqrt{2}-1} \right)^{1/2} \frac{1}{\sqrt{\varepsilon}} P_\sigma(\mathcal{E}), \end{aligned}$$

and the lemma is proved with $c' = c/5$. $\square$

Proof of Theorem 2. Assume that $u_1, \dots, u_s$ are arbitrary vectors in $\mathbf{R}^n$ with $|u_j|_n \leq 1$, $j = 1, \dots, s$. Set $v_j = u_j + e_{j+n}$, $j = 1, \dots, s$, where $\{e_i\}_{i \leq n+s}$ is the canonical orthonormal basis in $\mathbf{R}^{n+s}$. Then, the v_j 's are linearly independent vectors in $\mathbf{R}^{n+s}$, of Euclidean norm at most $\sqrt{2}$, and if

$$\mathcal{E}^* = \left\{ (\delta_j)_{j \leq s} : \left| \sum_{j \leq s} \delta_j v_j \right|_{n+s} \leq 1 \right\},$$

Lemma 4 implies that, given $\varepsilon \in (0, 1)$, there exists $\sigma \subseteq S$, $|\sigma| \geq (1 - \varepsilon)s$, for which

$$P_\sigma(\mathcal{E}^*) \supseteq c'' \sqrt{\varepsilon} D_\sigma$$

with $c'' = c/\sqrt{2}$, c the constant from Lemma 4. Since

$$\left| \sum_{j \leq s} \delta_j v_j \right|_{n+s}^2 = \left| \sum_{j \leq s} \delta_j u_j \right|_n^2 + \sum_{j \leq s} \delta_j^2,$$

we readily see that

$$P_\sigma(\mathcal{E}) \supseteq c'' \sqrt{\varepsilon} D_\sigma$$

and Theorem 2 is proved. $\square$

3. PROOF OF THEOREM 1 WITH $d = 1$

For the proof of the proportional Dvoretzky-Rogers factorization result, we shall combine Theorem 2 with the method used in [S-T]: Let $X = (\mathbf{R}^n, \|\cdot\|)$ be an n -dimensional normed space and $\varepsilon \in (0, 1)$ be given. Without loss of generality, we may assume that the ellipsoid of minimal volume containing the unit ball B_X of X is the Euclidean unit ball D . By John's theorem [J], $D \subseteq \sqrt{n} B_X$. We can also find contact points y_i , $i \leq N$, $\|y_i\|_X = \|y_i\|_n = 1$, $N = O(n^2)$, and positive real numbers μ_i , $i \leq N$, such that the following representation of the identity holds: for every $x \in \mathbf{R}^n$, $x = \sum_{i \leq N} \mu_i \langle x, y_i \rangle y_i$. Now, if s is the smallest integer $\geq (1 - \frac{\varepsilon}{2})n$, we can choose $x_1, \dots, x_s$ among the y_i 's so that:

Lemma 5 ([S-T]). $\text{dist}(x_i, \text{span}\{x_j, j \neq i\}) \geq \sqrt{\frac{\varepsilon}{2}}$, $i = 1, \dots, s$.

Hence, there exist v_j , $j \leq s$, in $\text{span}\{x_i, i \leq s\}$ satisfying

- (i) $|v_j|_n \leq \sqrt{2/\varepsilon}$, $j = 1, \dots, s$,
- (ii) $\langle x_i, v_j \rangle = \delta_{ij}$, $i, j = 1, \dots, s$.

Set $u_j = \sqrt{\varepsilon/2} v_j$ and define $\mathcal{E} = \{(\delta_j)_{j \leq s} : |\sum_{j \leq s} \delta_j u_j|_n \leq 1\}$. From Theorem 2 we obtain $\sigma \subseteq S$, $|\sigma| \geq (1 - \frac{\varepsilon}{2})s$, with $P_\sigma(\mathcal{E}) \supseteq c\sqrt{\varepsilon} D_\sigma$. Then $|\sigma| \geq (1 - \varepsilon)n$, and for any choice of scalars $\mathbf{t} = (t_i)_{i \in \sigma}$ we have

$$|\mathbf{t}|^2 = \sum_{i \in \sigma} t_i^2 = \left\langle \sum_{i \in \sigma} t_i x_i, \sum_{j \in \sigma} t_j v_j \right\rangle = \sqrt{\frac{2}{\varepsilon}} \left\langle \sum_{i \in \sigma} t_i x_i, \sum_{j \in \sigma} t_j u_j \right\rangle.$$

We can extend $(\frac{c\sqrt{\varepsilon}}{|\mathbf{t}|} t_j)_{j \in \sigma}$ to a vector $(\delta_j)_{j \leq s}$ in $\mathcal{E}$. Hence,

$$|\mathbf{t}|^2 = \sqrt{\frac{2}{\varepsilon}} \frac{|\mathbf{t}|}{c\sqrt{\varepsilon}} \left\langle \sum_{i \in \sigma} t_i x_i, \sum_{j \leq s} \delta_j u_j \right\rangle \leq \frac{c'}{\varepsilon} |\mathbf{t}| \left| \sum_{i \in \sigma} t_i x_i \right|_n$$

and, since $|\cdot|_n \leq \|\cdot\|_X$ and the x_i 's are of $\|\cdot\|$ -norm one, we have

$$\left(\sum_{i \in \sigma} t_i^2 \right)^{1/2} \leq \frac{c'}{\varepsilon} \left| \sum_{i \in \sigma} t_i x_i \right|_n \leq \frac{c'}{\varepsilon} \left\| \sum_{i \in \sigma} t_i x_i \right\|_X \leq \frac{c'}{\varepsilon} \sum_{i \in \sigma} |t_i|.$$

Defining $\beta: l_1^{|\sigma|} \rightarrow X$ with $\beta(e_i) = x_i$, $i \in \sigma$, and $\alpha: X \rightarrow l_2^{|\sigma|}$ with $\alpha = TP_\sigma$ where P_σ is the orthogonal projection of X onto $\text{span}\{x_i, i \in \sigma\}$ and $Tx_i = e_i$, we have a factorization $i_{1,2} = \alpha \circ \beta$ of the identity $i_{1,2}: l_1^{|\sigma|} \rightarrow l_2^{|\sigma|}$ with $\|\alpha\| \cdot \|\beta\| \leq c'/\varepsilon$. By duality and by the extension property of l_∞^n , this is then equivalent to the assertion of the theorem.

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