WEIGHTED INEQUALITIES FOR THE MAXIMAL GEOMETRIC MEAN OPERATOR

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ABSTRACT. For nonnegative Borel measures μ on R^1 and for the maximal geometric mean operator G_f , we characterize the weight pairs (w, v) for which G_f is of weak type (p, p) and of strong type (p, p) , $0 < p < \infty$. No doubling conditions are needed. We also note that a previously published different characterization for the strong type inequality for G_f has an incorrect proof.

1. INTRODUCTION

Let μ be a nonnegative Borel measure on R^1 . The maximal geometric mean operator is (see X. Shi [7])

$$G_f(x) = \sup \exp \left[\frac{1}{|I|} \int_I \log(|f(y)|) d\mu(y) \right],$$

where the sup is taken over all intervals I in R^1 containing x such that the integral is defined and $0 < |I| = \mu(I) < \infty$; if no such I exists we take $G_f(x) = 0$.

If $0 < p < \infty$, $w = v$ and $d\mu$ is Lebesgue measure on R^1 , then the inequality

$$(1) \quad \int G_f(x)^p w(x) d\mu(x) \leq C_1 \int |f(x)|^p v(x) d\mu(x)$$

for all f in $L^p(vd\mu)$ is equivalent to the requirement that

$$(2) \quad \left(\frac{1}{|I|} \int_I w d\mu \right) \left(\exp \left[\frac{1}{|I|} \int_I \log(1/v) d\mu \right] \right) \leq C_2$$

for all intervals I (see [7]) and is also equivalent to the weak type inequality

$$(3) \quad \int_{\{x|G_f(x)>\lambda\}} w d\mu \leq \frac{C_3}{\lambda^p} \int |f(x)|^p v(x) d\mu(x)$$

for all f in $L^p(vd\mu)$ (see [3]). In [3] it is stated that in spaces of homogeneous type condition (2) implies inequality (1) for general not necessarily equal v and w provided that μ and $vd\mu$ satisfy doubling conditions. The proof given in [3], however, is incorrect. They state on page 71 that "it is easy to see that" if $f_0(x) = f(x)$ on the set where $|f(x)| \leq \alpha/2$ and 0 elsewhere and $f_1(x) = f(x) - f_0(x)$, then $G_f(x) \leq G_{f_1}(x) + \alpha/2$. A simple counterexample is $f(x) = \chi_{(0,1]}(x, x) + e^n \chi_{(1,\infty)}(x, x)$ and $\alpha = 2$; for $0 < x < 1$ we have $G_f(x) = e^n$ and $G_{f_1}(x) = 0$. We

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will show in Theorem 1 that in general a stronger condition than (2) is needed to imply (1).

It might seem that since condition (2) implies (3) for all p , the Marcinkiewicz interpolation theorem would show that (2) implies (1). The Marcinkiewicz theorem cannot be applied here, however, because the operator is not quasilinear. This is easily seen in R^1 with $d\mu = dx$ by taking E to be a set such that $E \cap I$ and $E^c \cap I$ both have positive measure for every interval I . Then with $f = \chi_E$ and $g = \chi_{E^c}$ we have G_{f+g} equal to 1 for all x while G_f and G_g are 0 for all x .

2. THE MAIN THEOREMS

Theorem 1. *Let μ be a nonnegative Borel measure on R^1 , $0 < p < \infty$, and let w and v be two μ measurable weight functions. Then inequality (1) holds for all f in $L^p(vd\mu)$ if and only if*

$$(4) \quad \int_I G_{(v^{-1}\chi_I)}(x)w(x) d\mu(x) \leq C_4|I|$$

for every interval I .

Theorem 2. *Let w, v, μ and p be as in Theorem 1. Then inequality (3) holds for all f in $L^p(vd\mu)$ if and only if (2) holds for every interval I .*

From the proof of X. Shi [7] we can easily see that (4) is equivalent to (2) if $w = v$. This is also a simple consequence of Theorem 1 and well-known facts by the following reasoning. If $w = v$ satisfies (2), then by Theorem 1, page 254 of [2], w satisfies the condition A_∞ . By the Theorem on page 104 of [4] there is a $p > 1$ such that w satisfies A_p , and since $G_f(x) \leq Mf(x)$, where M denotes the Hardy-Littlewood maximal operator, we have (1) by Theorem 2, page 216 of [5]. Theorem 1 then shows that w satisfies (4).

Without the assumption $w = v$ conditions (2) and (4) are not equivalent; in section 6 we give an example of a pair of functions v, w that satisfy (2) but do not satisfy (4).

These theorems can be viewed as limiting cases of known results about the Hardy-Littlewood maximal operator M . This is based on the equality

$$\lim_{p \rightarrow \infty} \left(\frac{1}{|I|} \int_I v^{-1/p} d\mu \right)^p = \exp \left[\frac{1}{|I|} \int_I \log(1/v) d\mu \right],$$

which is valid if the left side is finite (see [1], [2]), and the consequence that $G_f(x) = \lim_{p \rightarrow \infty} [M(f^{1/p})]^p$ for suitably restricted f . Theorem 1 is a limiting case of the fact from Theorem B, p. 7 of [6] that $\int [M(f^{1/p})]^p w d\mu \leq C \int |f|v d\mu$ if and only if for every interval I $\int_I [M(\chi_I v^{-1/(p-1)})]^p w d\mu \leq C \int_I v^{-1/(p-1)} d\mu$. Theorem 2 is a limiting case of the fact from Theorem 6, p. 219 of [5] that

$$\int_{\{x | [M(f^{1/p})]^p > \lambda\}} w d\mu \leq \frac{C}{\lambda} \int |f|v d\mu$$

if and only if for every interval I

$$\left(\frac{1}{|I|} \int_I w d\mu \right) \left(\frac{1}{|I|} \int_I v^{-1/(p-1)} d\mu \right)^{p-1} \leq C.$$

3. COVERING LEMMAS

Our proofs of the theorems are based on the following lemmas. Lemma 1 is the covering idea used in the proof of Theorem (a), p. 1231 of [8].

Lemma 1. *If ν is a Borel measure, E is a subset of R^1 , B is a collection of intervals that cover E and there is a constant C such that $0 < \nu(I) \leq C$ for every I in B , then there is a disjoint sequence $\{I_j\}$ of intervals in B such that $\nu(E) \leq \sum 5\nu(I_j)$.*

To prove this each I_j is chosen to be disjoint from $I_1, \dots, I_{j-1}$ and with $\nu(I_j) > (1/2) \sup \nu(I)$, where the sup is taken over all I 's in B disjoint from $I_1, \dots, I_{j-1}$. If $\sum \nu(I_j) = \infty$, there is nothing more to prove. Therefore, assume that $\sum \nu(I_j) < \infty$. For each I_j define I_j^* to be the union of all intervals H in B that intersect I_j and have $\nu(H) < 2\nu(I_j)$. Then $\nu(I_j^*) \leq 5\nu(I_j)$ is immediate. Furthermore, if x is in E , let J be a member of B that contains x . Since $\lim_{j \rightarrow \infty} \nu(I_j) = 0$, there is a first member I_j of the sequence that intersects J . Then by the selection procedure $\nu(I_j) > (1/2)\nu(J)$ and by its definition $J \subset I_j^*$. From this $E \subset \bigcup I_j^*$. Therefore, $\nu(E) \leq \sum \nu(I_j^*) \leq \sum 5\nu(I_j)$. This completes the proof of Lemma 1.

Lemma 2. *If E is a subset of R^1 and B is a collection of closed intervals of positive length that cover E , then E is covered by a countable subcollection of B .*

To prove this let A be the set of points in E that occur only as left ends of intervals in B . For every x in A there is an interval in B that contains it, and this interval contains no other points of A in its interior. Therefore, A is countable and is consequently covered by a countable subcollection from B . The same reasoning applies to the points that occur only as right end points. The other points of E are covered by the open interiors of the intervals in B , and there is a countable subcollection that covers them by Lindelöf's theorem.

4. PROOF OF THEOREM 1

The proof that (1) implies (4) is trivial by taking $f(x) = v(x)^{-1/p} \chi_I(x)$.

For the proof that (4) implies (1) we need only consider the case $p = 2$ since $G_f(x)^p = G_{|f|^p}(x)$. It is enough to prove the result for bounded functions f with support E having finite μ measure and $\int |f|^2 v d\mu < \infty$. For these f 's it is enough to consider v 's with positive lower bound by the following reasoning. Given arbitrary $v(x)$, let $v_n(x)$ be the larger of $1/n$ and $v(x)$. Then v_n will still satisfy (4) with the same value of C_4 . It will appear in the proof that the constant C_1 depends only on C_4 . Therefore, (1) will be valid for all v_n 's with fixed C_1 , and the restriction on f insures that the right side of (1) converges properly as $n \rightarrow \infty$.

Having fixed such an f and v , we define a function F on intervals I in R^1 by

$$F(I) = \exp \left[\frac{1}{|I|} \int_I \log(|f(y)|) d\mu(y) \right]$$

for intervals with $0 < |I| < \infty$ for which the integral is defined and $F(I) = 0$ on other intervals. For each x for which $G_f(x) > 0$ we will now construct a closed interval I_x containing x such that $F(I_x) > G_f(x)/2$ and so that the end points of I_x are measurable functions of x .

To do this let E_k be the set of x for which $2^k < G_f(x) \leq 2^{k+1}$, and let C_k consist of all closed intervals I with $2^k < F(I) \leq 2^{k+1}$. Let A be the points a in E_k with $[a, a] \in C_k$. The set A is countable since if a is in A , $[[a, a]] > 0$ and a lies in the

support of f which has been assumed to have finite measure. Therefore, there is a countable subcollection of C_k that covers A . If $x \in E_k \cap A^c$ there is an interval I containing x with $F(I) > 2^k$ and I must have positive length since $[x, x]$ is not in C_k . If one or both ends of I are open, a closed interval J of positive length can be found inside I that contains x and has $F(J) > 2^k$. Therefore, the intervals of positive length in C_k cover $E_k \cap A^c$, and by Lemma 2 there is a countable subcollection that covers this set. Let $\{J_n\}$ be a sequence of closed intervals in C_k that cover E_k . For each x in E_k define I_x to be the first J_n that contains x . Then for x in E_k we have $F(I_x) > 2^k \geq G_f(x)/2$ and the ends of I_x are measurable functions because the set where $I_x = J_n$ is the measurable set $J_n \cap E_k \cap \bigcap_{m=1}^{n-1} J_m^c$.

Then if B is the set where $G_f(x) > 0$,

$$\begin{aligned}
 \int G_f(x)^2 w(x) d\mu(x) &< \int_B 4 \left[\exp \left(\frac{1}{|I_x|} \int_{I_x} \log(|f|) d\mu \right) \right]^2 w(x) d\mu(x) \\
 (5) \qquad &= 4 \int_B \left[\exp \left(\frac{1}{|I_x|} \int_{I_x} \log(|f|v^{1/2}) d\mu \right) \right]^2 \\
 &\quad \times \exp \left(\frac{1}{|I_x|} \int_{I_x} \log \left(\frac{1}{v} \right) d\mu \right) w(x) d\mu(x).
 \end{aligned}$$

Some justification is needed for this equality since in general an equality of the form

$$\exp \left[\int_I \log(g) d\mu \right] = \exp \left[\int_I \log(gh) d\mu \right] \exp \left[\int_I \log(1/h) d\mu \right]$$

can fail if the right side has the form $0 \cdot \infty$. This does not happen here because the restrictions on f and v insure that $\int_{I_x} \log(|f|) d\mu$, $\int_{I_x} \log(|f|v^{1/2}) d\mu$ and $\int_{I_x} \log(1/v) d\mu$ are all finite. By Jensen's inequality (5) is bounded by

$$(6) \qquad 4 \int_B \left[\frac{1}{|I_x|} \int_{I_x} |f|v^{1/2} d\mu \right]^2 \exp \left[\frac{1}{|I_x|} \int_{I_x} \log \left(\frac{1}{v} \right) d\mu \right] w(x) d\mu(x).$$

Define the measure U by $dU(x) = w(x) \exp(\frac{1}{|I_x|} \int_{I_x} \log(\frac{1}{v}) d\mu) d\mu(x)$, and let T be the linear operator defined by $Tg(x) = \frac{1}{|I_x|} \int_{I_x} g d\mu$. Then T is clearly bounded from $L^\infty(d\mu)$ to $L^\infty(dU)$. We shall show that T is bounded from $L^1(d\mu)$ to weak $L^1(dU)$. This will be sufficient since these imply that T is bounded from $L^2(d\mu)$ to $L^2(dU)$ and, therefore, that (6) equals

$$4 \int_B T(fv^{1/2})^2 dU \leq C \int |fv^{1/2}|^2 d\mu = C \int |f|^2 v d\mu.$$

To prove the weak type assertion for T fix g in $L^1(d\mu)$ and $\lambda > 0$, let $D = \{Tg(x) > \lambda\}$, and let $E = \bigcup_{x \in D} I_x$. Since the set of all I_x 's is countable, E can be written as the countable union of its connected components $\{J_n\}$. Note that if x is in $D \cap J_n$, then $I_x \subset J_n$. We have, therefore,

$$\begin{aligned}
 \int_D dU &= \sum_n \int_{D \cap J_n} \exp \left[\frac{1}{|I_x|} \int_{I_x} \log \left(\left| \frac{1}{v} \right| \right) d\mu \right] w(x) d\mu(x) \\
 &\leq \sum_n \int_{J_n} \exp \left[\frac{1}{|I_x|} \int_{I_x} \log \left(\left| \frac{1}{v} \right| \chi_{J_n} \right) d\mu \right] w(x) d\mu(x).
 \end{aligned}$$

By condition (4) this is bounded by

$$(7) \quad C_4 \sum_n |J_n|.$$

Now each J_n is the union of I_x 's with x in D . From the fact that x is in D we have $0 < |I_x| < \frac{1}{\lambda} \int_{I_x} g \, d\mu \leq \frac{1}{\lambda} \|g\|_1$. We can, therefore, apply Lemma 1 to each J_n . This will produce disjoint sequences of intervals $I_{n,k} \subset J_n$, and since the $I_{n,k}$'s are I_x 's with $x \in D$,

$$(8) \quad \lambda < \frac{1}{|I_{n,k}|} \int_{I_{n,k}} |g| \, d\mu.$$

Furthermore, since the J_n 's are disjoint, all the intervals $I_{n,k}$ are disjoint. Using these two facts and Lemma 1 shows that (7) has the bound

$$5C_4 \sum_{n,k} |I_{n,k}| < \sum_{n,k} \frac{5}{\lambda} C_4 \int_{I_{n,k}} |g| \, d\mu \leq \frac{5}{\lambda} C_4 \|g\|_1.$$

This completes the proof of Theorem 1.

5. PROOF OF THEOREM 2

The proof that (3) implies (2) is trivial by taking $f(x) = v(x)^{-1/p} \chi_I(x)$ and $\lambda = (1/2) \exp(\frac{1}{|I|} \int_I \log(v^{-1/p}) \, d\mu)$.

For the proof that (2) implies (3) we need only consider the case $p = 1$. As in the case of the proof of Theorem 1 we need only consider functions f that are bounded with support of finite μ measure and $\int |f|v \, d\mu < \infty$ and weights v with a positive lower bound. Given such functions f and v and a $\lambda > 0$, define an operator M_f by

$$M_f(x) = \sup_I \frac{\int_I |f(y)|v(y) \, d\mu(y)}{\int_I w(t) \, d\mu(t)}$$

with the sup taken over all intervals I containing x with the quotient taken as 0 in ambiguous cases. By (2), Jensen's inequality and the restrictions on f and v

$$(9) \quad \begin{aligned} G_f(x) &= \sup_I \exp \left(\left[\frac{1}{|I|} \int_I \log(|f|v) \, d\mu \right] \left[\frac{1}{|I|} \int_I \log \left(\frac{1}{v} \right) \, d\mu \right] \right) \\ &\leq \sup_I C_2 \frac{\frac{1}{|I|} \int_I |f|v \, d\mu}{\frac{1}{|I|} \int_I w \, d\mu} = C_2 M_f(x). \end{aligned}$$

Let E be the set where $M_f(x) > \lambda/C_2$, let D be the set where $M_f(x) = \infty$ and let C be the collection of intervals I such that $\int_I |f|v \, d\mu > 0$ and $\int_I w \, d\mu = 0$. Let A be the sets of points a such that $[a, a] \in C$. The assumption that $\int |f|v \, d\mu < \infty$ implies that A is countable. The set $D \cap A^c$ is covered by intervals in C with positive length. Therefore, by Lemma 2 and the countability of A , D has a countable covering $\{I_k\}$ of intervals in C . From this $\int_D w \, d\mu \leq \sum_k \int_{I_k} w \, d\mu = 0$. This and (9) show that

$$(10) \quad \int_{\{G_f(x) > \lambda\}} w \, d\mu \leq \int_E w \, d\mu = \int_{E \cap D^c} w \, d\mu.$$

Now let B consist of all intervals I with $\int_I |f|v \, d\mu > \frac{\lambda}{C_2} \int_I w \, d\mu > 0$. Then B covers $E \cap D^c$ and we can apply Lemma 1 with $d\nu = w \, d\mu$. The result is a disjoint

sequence $\{J_n\}$ from B and

$$(11) \quad \int_{E \cap D^c} w \, d\mu \leq 5 \sum_n \int_{J_n} w \, d\mu < \frac{5C_2}{\lambda} \sum_n \int_{J_n} |f|v \, d\mu \leq \frac{5C_2}{\lambda} \int |f|v \, d\mu.$$

Combining (10) and (11) completes the proof of Theorem 2.

6. AN EXAMPLE

In this section we derive a pair (w, v) of functions that satisfy (2) but do not satisfy (1). It is also easily seen that they do not satisfy (4).

Let $d\mu(x) = dx$,

$$v(x) = \begin{cases} \exp\left[-\frac{1}{x(\log x)^2}\right], & x \in (0, e^{-2}], \\ \infty, & \text{elsewhere,} \end{cases}$$

and

$$w(x) = \chi_{(0, e^{-2}]}(x) \frac{d}{dx} \left[x \exp\left(\frac{1}{x \log x}\right) \right].$$

Note that

$$w(x) = \chi_{(0, e^{-2}]}(x) \left(1 - \frac{1}{x \log x} - \frac{1}{x(\log x)^2} \right) \exp\left(\frac{1}{x \log x}\right)$$

so that for x in $(0, e^{-2}]$

$$(12) \quad w(x) \leq \frac{-2}{x \log x} \exp\left(\frac{1}{x \log x}\right)$$

and

$$(13) \quad w(x) \geq \frac{-1}{2x \log x} \exp\left(\frac{1}{x \log x}\right).$$

To prove (2) we need only consider intervals $I \subset [0, e^{-2}]$ since the second factor in (2) is 0 if $I \not\subset [0, e^{-2}]$. We will prove that (2) holds for $I = (a, b)$ in two cases: $a < 2b/3$ and $a \geq 2b/3$.

For $a < 2b/3$

$$(14) \quad \frac{1}{|I|} \int_I w(x) \, dx \leq \frac{3}{b} \int_0^b w(x) \, dx = 3 \exp\left(\frac{1}{b \log b}\right),$$

and since $\log(\frac{1}{v}) = \frac{1}{x(\log x)^2}$ is decreasing on $(0, e^{-2}]$,

$$\frac{1}{|I|} \int_I \log\left(\frac{1}{v(x)}\right) \, dx < \frac{1}{b} \int_0^b \log\left(\frac{1}{v(x)}\right) \, dx = \frac{-1}{b \log b}.$$

Therefore,

$$(15) \quad \exp\left[\frac{1}{|I|} \int_I \log\left(\frac{1}{v(x)}\right) \, dx\right] < \exp\left(\frac{-1}{b \log b}\right).$$

Combining (14) and (15) gives (2) with $C_2 = 3$ for this case.

For $a \geq 2b/3$ use (12) and the fact that the right side of (12) is increasing on $(0, e^{-2}]$ to see that

$$(16) \quad \frac{1}{|I|} \int_I w(x) \, dx < \frac{-2}{b \log b} \exp\left(\frac{1}{b \log b}\right).$$

Next, since $\frac{1}{x(\log x)^2}$ decreases on $(0, e^{-2}]$,

$$(17) \quad \frac{1}{|I|} \int_I \log \left(\frac{1}{v} \right) dx = \frac{1}{b-a} \int_a^b \frac{dx}{x(\log x)^2} < \frac{1}{a(\log a)^2}.$$

Now for $2b/3 \leq a < b \leq e^{-2}$

$$\frac{1}{a(\log a)^2} < \frac{-1}{2a \log a} < \frac{-1}{(4/3)b \log b}.$$

Using this, (16) and (17) gives

$$\left[\frac{1}{|I|} \int_I w(x) dx \right] \left[\exp \left(\frac{1}{|I|} \int_I \log \left(\frac{1}{v(x)} \right) dx \right) \right] < \frac{-2}{b \log b} \exp \left(\frac{1}{4b \log b} \right).$$

Since $8y \exp(-y) \leq 8/e$ for y in $(0, \infty)$, the right side is bounded by $8/e$. This completes the proof of (2) with $C_2 = 3$ for this pair.

To show that (1) is false for this v and w take $f(x) = v(x)^{-1/p} \chi_{(0, e^{-2})}(x)$. The integral on the right side of (1) is e^{-2} . Now

$$G_f(x) \geq \exp \left(\frac{1}{x} \int_0^x \log f(t) dt \right) = \exp \left(\frac{-1}{px \log x} \right).$$

This and (13) show that the left side of (1) is bounded below by $\int_0^{e^{-2}} \frac{-dx}{2x \log x} = \infty$. This computation with $p = 1$ also shows directly that (4) fails for this pair.

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