ON RESULTS OF CZUBIAK-GUNDERSEN AND OSGOOD-YANG

HONG-XUN YI

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ABSTRACT. This paper studies the problem of uniqueness of entire functions that share real zeros and real ones. An example is provided to show that a result of Czubiak and Gundersen is not completely correct. Results in this paper correct the result of Czubiak and Gundersen, and also correct a result of Osgood and Yang.

1. INTRODUCTION AND MAIN RESULTS

We say that two entire functions $f(z)$ and $g(z)$ share the value c provided that $f(z) = c$ if and only if $g(z) = c$. We distinguish between sharing a value CM (counting multiplicities) and IM (ignoring multiplicities).

Let $\mathcal{K}$ be the class of all nonconstant entire functions which have only real zeros and real ones. T. P. Czubiak and G. G. Gundersen obtained the following result:

Theorem A ([1, Theorem]). *If f and g are in $\mathcal{K}$ and share 0 and 1 IM, then we necessarily have one of the following six cases where $a \neq 0$ and b are real constants:*

- (i) $f \equiv g$;
- (ii) $f(z) = \sin^2(az + b)$ and $g(z) = -i \sin(az + b)e^{i(az+b)}$;
- (iii) $f(z) = \frac{\sin^2(p(az + b))}{\sin^2(az + b)}$ and $g(z) = \frac{\sin(p(az + b))}{\sin(az + b)}e^{i(p-1)(az+b)}$
 for $p = -2$ and for $p = -3$;
- (iv) $f(z) = 1 - \frac{(g(z) - 1)^2}{e^{2pi(az+b)}}$ and $g(z) = \frac{\sin(p(az + b))}{\sin(az + b)}e^{i(p-1)(az+b)}$
 for $p = 3$ and for $p = 4$.

From Theorem A Czubiak and Gundersen obtained the following result.

Theorem B ([1, Corollary 1]). *If f and g are in $\mathcal{K}$ and share 0 and 1 CM, then $f \equiv g$.*

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The above results of Czubiak and Gundersen, however, are not true for $f(z) = (\frac{1}{2} + \text{id})(1 - e^{iz})$ and $g(z) = (\frac{1}{2} - \text{id})(1 - e^{-iz})$, where d is a real constant.

In this note, we prove the following theorems which are corrections of Theorem A and Theorem B.

Theorem 1. *If f and g are in $\mathcal{K}$ and share 0 and 1 IM, then f and g must satisfy exactly one of the following relations:*

$$(i) \quad f \equiv g;$$

$$(ii) \quad \begin{aligned} f(z) &= \frac{\sin(az+b)}{\sin(b-c)} e^{i(az+c)} = \left(\frac{1}{2} + \text{id}\right) (1 - e^{2i(az+b)}), \\ g(z) &= \frac{\sin(az+b)}{\sin(b-c)} e^{-i(az+c)} = \left(\frac{1}{2} - \text{id}\right) (1 - e^{-2i(az+b)}), \end{aligned}$$

where a, b, c, d are real constants, and $a \neq 0$, $\sin(b-c) \neq 0$, $d = \frac{1}{2} \cot(b-c)$;

$$(iii) \quad \begin{aligned} f(z) &= \frac{\sin(p(az+b))}{\sin(az+b)} e^{i(p-1)(az+b)}, \\ g(z) &= \frac{\sin(p(az+b))}{\sin(az+b)} e^{-i(p-1)(az+b)}, \end{aligned}$$

where p is an integer, a, b are real constants, and $p \neq 0, 1$, $a \neq 0$;

$$(iv) \quad f(z) = \sin^2(az+b) \quad \text{and} \quad g(z) = -i \sin(az+b) e^{i(az+b)},$$

where $a \neq 0$ and b are real constants;

$$(v) \quad f(z) = \frac{\sin^2(p(az+b))}{\sin^2(az+b)} \quad \text{and} \quad g(z) = \frac{\sin(p(az+b))}{\sin(az+b)} e^{i(p-1)(az+b)},$$

for $p = -2$ and for $p = -3$, where $a \neq 0$, b are real constants;

$$(vi) \quad \begin{aligned} f(z) &= 1 - \frac{\sin^2((p-1)(az+b))}{\sin^2(az+b)}, \\ g(z) &= \frac{\sin(p(az+b))}{\sin(az+b)} e^{i(p-1)(az+b)}, \end{aligned}$$

for $p = 3$ and for $p = 4$, where $a \neq 0$, b are real constants.

Theorem 2. *If f and g are in $\mathcal{K}$ and share 0 and 1 CM, then f and g must satisfy exactly one of the relations (i), (ii), and (iii) of Theorem 1.*

2. SOME LEMMAS

In order to prove our theorems, we need the following lemmas.

Lemma 1 (cf. [2, Corollary of Theorem 2]). *If f is in $\mathcal{K}$, then the order of f is at most one.*

Lemma 2 (cf. [2, Proposition 1]). *If f is in $\mathcal{K}$, and f is not real for some real value of z , then f is necessarily of one of the following two forms:*

$$(i) \quad f(z) = \frac{\sin(az+b)}{\sin(b-c)} e^{i(az+c)}, \quad \sin(b-c) \neq 0,$$

where $a \neq 0$, b and c are real constants;

$$(ii) \quad f(z) = \frac{\sin(p(az+b))}{\sin(az+b)} e^{i(p-1)(az+b)},$$

where $p \neq 0, 1$ is an integer, and $a \neq 0$, b are real constants.

Lemma 3 ([3, Theorem 2]). *Let $Q_1(z)$ and $Q_2(z)$ be nonconstant polynomials of the same degree. If*

$$f(z) = \frac{e^{Q_1(z)} - 1}{e^{Q_2(z)} - 1}$$

is a nonconstant entire function, then

$$Q_1(z) = pQ_2(z) + 2q\pi i,$$

where $p \neq 0, 1$ and q are integers.

3. PROOF OF THEOREM 2

Since f and g are in $\mathcal{K}$, by Lemma 1 we have $\text{order}(f) \leq 1$ and $\text{order}(g) \leq 1$. From the fact that f and g share the values $0, 1$ CM, we obtain

$$(1) \quad \frac{f(z)}{g(z)} = e^{a_1 z + b_1}, \quad \frac{f(z) - 1}{g(z) - 1} = e^{a_2 z + b_2},$$

where a_1, a_2, b_1 , and b_2 are constants. Suppose that $f \not\equiv g$. Then from (1) we deduce that

$$(2) \quad f(z) = \frac{e^{a_2 z + b_2} - 1}{e^{a_3 z + b_3} - 1}, \quad g(z) = \frac{e^{-(a_2 z + b_2)} - 1}{e^{-(a_3 z + b_3)} - 1},$$

where $a_3 = a_2 - a_1, b_3 = b_2 - b_1$ are constants, and $e^{a_3 z + b_3} \not\equiv 1, a_2 \neq 0$.

Assume that $a_3 = 0$ and $e^{b_3} = K$. Then $K \neq 0, 1$. From (2) we have

$$(3) \quad f(z) = \frac{1}{K-1}(e^{a_2 z + b_2} - 1), \quad g(z) = \frac{K}{1-K}(e^{-(a_2 z + b_2)} - 1).$$

Hence, $z_1 = (2\pi i/a_2) - (b_2/a_2)$ and $z_2 = z_1 + (2\pi i/a_2)$ are two zeros of f . Since z_1 and z_2 are real, we obtain $\text{Re}(a_2) = \text{Re}(b_2) = 0$. Let $a = a_2/2i$ and $b = b_2/2i$; then $a \neq 0$ and b are real constants. Again from (3), we get

$$(4) \quad f(z) = \frac{1}{K-1}(e^{2i(az+b)} - 1), \quad g(z) = \frac{K}{1-K}(e^{-2i(az+b)} - 1).$$

Thus

$$(5) \quad f(z) - 1 = \frac{1}{K-1}(e^{2i(az+b)} - K).$$

Let $K_1 + iK_2 = \frac{1}{2} \log K$, where K_1 and K_2 are real constants. From (5), we obtain $z_0 = (K_2 - b)/a - iK_1/a$ is a one-point of f . Since z_0 is real, we obtain $K_1 = 0$. Let $c = b - K_2$; then c is a real constant. Again from (4), we have

$$f(z) = \frac{e^{2i(az+b)} - 1}{e^{2i(b-c)} - 1} = \frac{\sin(az+b)}{\sin(b-c)} e^{i(az+c)},$$

$$g(z) = \frac{e^{2i(b-c)}}{1 - e^{2i(b-c)}} (e^{-2i(az+b)} - 1) = \frac{\sin(az+b)}{\sin(b-c)} e^{-i(az+c)}.$$

Assume that $a_3 \neq 0$. Applying Lemma 3 to (2), we have

$$a_2 z + b_2 = p(a_3 z + b_3) + 2q\pi i,$$

where $p \neq 0, 1$ and q are integers. Thus

$$(6) \quad f(z) = \frac{e^{p(a_3z+b_3)} - 1}{e^{(a_3z+b_3)} - 1}, \quad g(z) = \frac{e^{-p(a_3z+b_3)} - 1}{e^{-(a_3z+b_3)} - 1}.$$

If $p \neq -1$, then $z_3 = 2m\pi i/pa_3 - b_3/a_3$ (m is an integer $\neq 0, \text{mod}(p)$) and $z_4 = z_3 + 2\pi i/a_3$ are two zeros of f . If $p = -1$, then $z_5 = \pi i/a_3 - b_3/a_3$ and $z_6 = z_5 + 2\pi i/a_3$ are two one-points of f . Since z_3, z_4, z_5 , and z_6 are real, we obtain $\text{Re}(a_3) = \text{Re}(b_3) = 0$. Let $a = a_3/2i$ and $b = b_3/2i$; then $a \neq 0$ and b are real constants. Again from (6), we get

$$f(z) = \frac{e^{2pi(az+b)} - 1}{e^{2i(az+b)} - 1} = \frac{\sin(p(az+b))}{\sin(az+b)} e^{i(p-1)(az+b)},$$

$$g(z) = \frac{e^{-2pi(az+b)} - 1}{e^{-2i(az+b)} - 1} = \frac{\sin(p(az+b))}{\sin(az+b)} e^{-i(p-1)(az+b)}.$$

This completes the proof of Theorem 2.

4. PROOF OF THEOREM 1

We note that in [1] in fact Czubiak and Gundersen proved the following results:

Theorem C. *If f and g are in $\mathcal{K}$ and share 0 and 1 IM and they are also both real on the real axis, then f and g share 0 and 1 CM.*

Theorem D. *If f and g are in $\mathcal{K}$ and share 0 and 1 IM and f is real for real z and g is not real for some real value of z , then f and g must satisfy exactly one of the following relations:*

$$(i) \quad f(z) = \sin^2(az+b) \quad \text{and} \quad g(z) = -i \sin(az+b) e^{i(az+b)},$$

where $a \neq 0$ and b are real constants;

$$(ii) \quad f(z) = \frac{\sin^2(p(az+b))}{\sin^2(az+b)} \quad \text{and} \quad g(z) = \frac{\sin(p(az+b))}{\sin(az+b)} e^{i(p-1)(az+b)},$$

for $p = -2$ and for $p = -3$, where $a \neq 0$, b are real constants;

$$(iii) \quad f(z) = 1 - \frac{(g(z)-1)^2}{e^{2pi(az+b)}} = 1 - \frac{\sin^2((p-1)(az+b))}{\sin^2(az+b)},$$

$$g(z) = \frac{\sin(p(az+b))}{\sin(az+b)} e^{i(p-1)(az+b)},$$

for $p = 3$ and for $p = 4$, where $a \neq 0$, b are real constants.

Using Lemma 2, we have the following result:

Theorem 3. *If f and g are in $\mathcal{K}$ and share 0 and 1 IM and neither f nor g is real for all real z , then f and g share 0 and 1 CM.*

Proof. Since f and g are in $\mathcal{K}$ and neither f nor g is real for all real z , f and g must have the forms in Lemma 2. Therefore, f and g have only simple zeros. To see that f and g have only simple one-points, one can consider $F = 1 - f$ and $G = 1 - g$ and use the same reasoning. Theorem 3 follows. $\square$

It is obvious that the functions $f(z)$ and $g(z)$ in the relations (ii) and (iii) of Theorem 1 are not real for all real z . From Theorem 2, Theorem C, and Theorem 3, we have the following results:

Theorem C'. *If f and g are in $\mathcal{K}$ and share 0 and 1 IM and they are also both real on the real axis, then $f \equiv g$.*

Theorem 3'. *If f and g are in $\mathcal{K}$ and share 0 and 1 IM and neither f nor g is real for all real z , then f and g must satisfy exactly one of the relations (i), (ii), and (iii) of Theorem 1.*

Now using Theorem C', Theorem 3', and Theorem D, we finally obtain the conclusion of Theorem 1.

5. CORRECTION OF A RESULT OF OSGOOD AND YANG

In [1], Czubiak and Gundersen used the following result obtained by C. F. Osgood and C. C. Yang:

Theorem E ([3, Theorem 3]). *If f and g are nonconstant entire functions of finite order and share 0 and 1 CM, then either $f \equiv g$ or*

$$f(z) = 1 + e^Q + \cdots + e^{(n-1)Q} = (1 - e^{nQ})/(1 - e^Q)$$

and

$$g(z) = 1 + e^{-Q} + \cdots + e^{-(n-1)Q} = (1 - e^{-nQ})/(1 - e^{-Q}),$$

or more generally

$$f(z) = (1 - e^{2\pi i P(Q)})/(1 - e^{2\pi i Q})$$

and

$$g(z) = (1 - e^{2\pi i P(Q)})/(e^{2\pi i(P(Q)-Q)} - e^{2\pi i P(Q)}),$$

where Q is a polynomial in z and $P(z)$ is a polynomial in z with rational numbers as the coefficients.

The above result of Osgood and Yang, however, is not true for

$$f(z) = \left(\frac{1}{2} + id\right)(1 - e^{iz}) \quad \text{and} \quad g(z) = \left(\frac{1}{2} - id\right)(1 - e^{-iz}),$$

where d is a real constant. The following result is a correction of Theorem E:

Theorem E'. *If f and g are nonconstant entire functions of finite order and share 0 and 1 CM, then f and g must satisfy exactly one of the following relations:*

$$(i) \quad f \equiv g;$$

$$(ii) \quad f(z) = \frac{1}{K-1}(e^{P(z)} - 1) \quad \text{and} \quad g(z) = \frac{K}{1-K}(e^{-P(z)} - 1),$$

where $K \neq 0, 1$ is a constant, and $P(z)$ is a nonconstant polynomial;

$$(iii) \quad \begin{aligned} f(z) &= (1 - e^{nQ(z)})/(1 - e^{Q(z)}), \\ g(z) &= (1 - e^{-nQ(z)})/(1 - e^{-Q(z)}), \end{aligned}$$

where $n \neq 0, 1$ is an integer, and $Q(z)$ is a nonconstant polynomial;

$$(iv) \quad \begin{aligned} f(z) &= (1 - e^{2\pi i P(Q(z))})/(1 - e^{2\pi i Q(z)}), \\ g(z) &= (1 - e^{-2\pi i P(Q(z))})/(1 - e^{-2\pi i Q(z)}), \end{aligned}$$

where $Q(z)$ is a nonconstant polynomial in z , and $P(z)$ is a polynomial in z with rational numbers as the coefficients, and $\deg(P) \geq 2$.

We may rewrite the functions $f(z)$ and $g(z)$ in the relation (iii) of Theorem E' in the following forms:

$$(iii)_a \quad \begin{aligned} f(z) &= 1 + e^{Q(z)} + \dots + e^{(n-1)Q(z)}, \\ g(z) &= 1 + e^{-Q(z)} + \dots + e^{-(n-1)Q(z)}, \end{aligned}$$

where $n \geq 2$;

$$(iii)_b \quad \begin{aligned} f(z) &= -e^{-mQ(z)} - e^{-(m-1)Q(z)} - \dots - e^{-Q(z)}, \\ g(z) &= -e^{mQ(z)} - e^{(m-1)Q(z)} - \dots - e^{Q(z)}, \end{aligned}$$

where $m = -n \geq 1$.

Next we proceed to prove Theorem E'.

By the assumption of Theorem E', we have two polynomials, $R(z)$ and $S(z)$, such that

$$(7) \quad f = ge^R \quad \text{and} \quad f - 1 = (g - 1)e^S.$$

Suppose that $f \neq g$. Then $e^S \neq 1$ and $e^{S-R} \neq 1$. Thus from (7) we obtain

$$(8) \quad \begin{aligned} f(z) &= (1 - e^{S(z)})/(1 - e^{S(z)-R(z)}), \\ g(z) &= (1 - e^{-S(z)})/(1 - e^{R(z)-S(z)}). \end{aligned}$$

We discuss the following two cases.

(A) Assume that $e^{S(z)-R(z)} \equiv K$, where K is a constant. Then $K \neq 0, 1$. Thus from (8) we have

$$f(z) = \frac{1}{K-1}(e^{P(z)} - 1) \quad \text{and} \quad g(z) = \frac{K}{1-K}(e^{-P(z)} - 1),$$

where $P(z) = S(z)$ is a nonconstant polynomial.

(B) Assume that $e^{S(z)-R(z)}$ is not a constant. In [3], in fact Osgood and Yang proved that in this case $f(z)$ and $g(z)$ must satisfy exactly one of the relations (iii) and (iv) of Theorem E'. Theorem E' is thus proved.

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DEPARTMENT OF MATHEMATICS, SHANDONG UNIVERSITY, JINAN, SHANDONG, 250100, PEOPLE'S REPUBLIC OF CHINA