FACTORIZATION THEOREMS FOR HARDY SPACES OF THE BIDISC, $0 < p \leq 1$

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ABSTRACT. A factorization theorem is proved in the Hardy spaces H^p of the bi-upper half plane, $0 < p \leq 1$. The proof is based on some fundamental work of Chang-Fefferman on atomic decompositions of H^p .

1. INTRODUCTION AND PRELIMINARIES

We are concerned with a factorization theorem which has been known (for $p = 1$) for the unit disc in $\mathbf{C}$ since the first part of the twentieth century. In 1976, Coifman, Rochberg and Weiss [4] extended it to the unit ball in $\mathbf{C}^n$. In 1992, Krantz and Li [6] proved that it holds on smoothly bounded strongly pseudoconvex domains for $0 < p \leq 1$. In this paper, we prove the analogous factorization theorem for the Hardy spaces H^p of the bi-upper half plane, $0 < p \leq 1$. A good reference for Hardy spaces is the book by Krantz [5, Chapter 8].

The following standard notation will be used: $\mathbf{R}$ denotes the real numbers; $\mathbf{C}$ denotes the complex numbers; $x = (x_1, x_2, \dots, x_n)$ denotes an element of $\mathbf{R}^n$.

As a consequence of the boundedness of the Hilbert transform on $L^q(\mathbf{R})$ for $1 < q < \infty$ (see [8, p. 38]) it follows that $S: L^q(\mathbf{R}) \rightarrow H^q(\mathbf{R}_+^2)$ is bounded, where S is the Szegő projection for the upper half plane:

$$Sf(z) = \frac{1}{\pi} \int_{\mathbf{R}} \frac{f(w)}{z - w} dw,$$

and $S(z, w) = \frac{1}{\pi} \frac{1}{z - w}$ is the Szegő kernel on the upper half plane $\mathbf{R}_+^2 := \{z = x + iy: x \in \mathbf{R}, y > 0\}$.

It is a simple matter to extend this result to higher dimensions.

Lemma 1.1. *If $q > 1$ and $n \geq 1$, then*

$$S: L^q(\mathbf{R}^n) \rightarrow H^q((\mathbf{R}_+^2)^n)$$

is bounded.

The following two lemmas will be useful.

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Lemma 1.2. Let $\psi \in L^2(\mathbf{R})$ with support $\subseteq [-l, l]$ and, for $z \in \mathbf{R}_+^2$ let

$$(1) \quad \phi(z) = \int_{\mathbf{R}} \psi(w) S(z, w) dw = \frac{1}{\pi} \int_{\mathbf{R}} \frac{\psi(w)}{z - w} dw.$$

Then $\phi \in H^2(\mathbf{R})$ and (identifying ϕ with its boundary values in $L^2(\mathbf{R})$)

$$(1) \quad \int |\phi(x)|^2 dx \leq \int |\psi(w)|^2 dw.$$

(2) If $|x| > 2l$ and $x \in \mathbf{R}$, then

$$|\phi(x)| \leq c \frac{\|\psi\|_1}{|x|} \quad a.e.$$

(3) If $\int_{\mathbf{R}} \psi = 0$ and $|x| > 2l$ and $x \in \mathbf{R}$, then

$$|\phi(x)| \leq \frac{cl\|\psi\|_1}{|x|^2} \quad a.e.$$

Proof. Part (1) of the lemma follows from the fact that ϕ is the image of ψ under the Szegő projection of $L^2(\mathbf{R})$ onto $H^2(\mathbf{R})$.

(2) If $|x| > 2l$, $x \in \mathbf{R}$, then for $w \in [-l, l]$,

$$|x| \leq |x - w| + |w| \leq |x - w| + l.$$

Therefore,

$$|x - w| \geq |x| - l \geq |x| - \frac{|x|}{2} = \frac{|x|}{2},$$

and thus

$$|x - w|^{-1} \leq c|x|^{-1}.$$

So for a.e. x , we obtain from (1)

$$|\phi(x)| \leq c|x|^{-1} \int_{-l}^l |\psi(w)| dw \leq c|x|^{-1} \|\psi\|_1.$$

(3) If $|x| > 2l$, $x \in \mathbf{R}$, $\int \psi = 0$, then for a.e. x

$$\begin{aligned} \phi(x) &= c \int_{\mathbf{R}} \psi(w) S(x, w) dw \\ &= c \int_{\mathbf{R}} \psi(w) [S(x, w) - S(x, 0)] dw \\ &= c \int_{-l}^l \frac{\psi(w)w}{(x - w)x} dw. \end{aligned}$$

As in the argument of part (2), we see that

$$|\phi(x)| \leq cl|x|^{-2} \int_{-l}^l |\psi(w)| dw = cl|x|^{-2} \|\psi\|_1.$$

Lemma 1.3. If $\alpha > 0$, then

$$\int_{\mathbf{R} \setminus [-2l, 2l]} l^\alpha |x|^{-(1+\alpha)} dx = c_\alpha = \alpha^{-1} 2^{-(\alpha+1)}.$$

We shall conclude this section with a discussion of the atomic decomposition of Chang and Fefferman for H^p , $0 < p \leq 1$.

In one variable (see [3], [7]) if $f \in H^1(\mathbf{R}^1)$, then $f(x)$ can be written as

$$f(x) = \sum \lambda_k a_k(x)$$

where $\sum |\lambda_k| \leq C\|f\|_{H^1}$ and $a_k(x)$ are particularly simple functions called “atoms”.

An analogous decomposition holds for functions f defined on $\mathbf{R}^2$ which are boundary values of functions in $H^1(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$, where $\mathbf{R}_+^2$ is the upper half plane.

In what follows, we shall deal exclusively with the domain $\mathbf{R}_+^2 \times \mathbf{R}_+^2$ and its Šilov boundary $\mathbf{R}^2$. A point of $\mathbf{R}_+^2 \times \mathbf{R}_+^2$ will be denoted by $z = (z_1, z_2)$ where $z_j = x_j + iy_j$ and $x_j \in \mathbf{R}$, $y_j > 0$, $j = 1, 2$.

Now, we shall introduce some notation: Let $\psi \in C^1(\mathbf{R})$ be supported on $[-1, 1]$ with ψ even and $\int_{-1}^1 \psi(x) dx = 0$. If $y > 0$, $\psi_y(x) = (1/y)\psi(x/y)$ and if $y = (y_1, y_2)$ and $x = (x_1, x_2) \in \mathbf{R}^2$, then $\psi_y(x) = \psi_{y_1}(x_1)\psi_{y_2}(x_2)$. If f is a function defined on $\mathbf{R}$, then we define $f(x, y) := f * \psi_y(x)$; if $x = (x_1, x_2) \in \mathbf{R}^2$, we denote $\Gamma(x) := \Gamma(x_1) \times \Gamma(x_2)$, where $\Gamma(x_j) := \{(t_j, y_j) \in \mathbf{R}^2 : |x_j - t_j| < y_j\}$, $j = 1, 2$.

Given a function f on $\mathbf{R}^2$, we define its double Square function by

$$Q^2(f)(x) := \int \int_{\Gamma(x)} |f(t, y)|^2 \frac{dt dy}{y_1^2 y_2^2}.$$

It is a fact that for $1 < p < \infty$

$$\|Q(f)\|_p \leq c_p \|f\|_p.$$

We may also define functions in $H^p(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$, $0 < p < \infty$, as those functions f with $Q(f) \in L^p(\mathbf{R}^2)$ and define $\|f\|_{H^p} = \|Q(f)\|_p$.

The following definition is due to Chang and Fefferman.

Definition 1.4. A p -atom ($0 < p \leq 1$) is a function $a(x)$, $x = (x_1, x_2) \in \mathbf{R}^2$, defined on $\mathbf{R}^2$ whose support is contained in some open set Ω of finite measure such that

- (1) $\|a\|_{L^2} \leq |\Omega|^{1/2-1/p}$.
- (2) a can be further decomposed into “elementary particles” a_R as follows:
 - (a) $a = \sum_{R \subset \Omega} a_R$, where a_R is supported in a rectangle $R \subseteq \Omega$ (say, $R = I_1 \times I_2$) and the R in the sum have the property that no one R is contained in the triple of any other.
 - (b) $\int_{I_1} a_R(x_1, x'_2) x_1^k dx_1 = 0 = \int_{I_2} a_R(x'_1, x_2) x_2^k dx_2$ for each $x'_j \in I_j$, $j = 1, 2$, and $\forall k = 1, 2, \dots, k(p)$, where $k(p)$ is an integer depending on p , $k(p) \leq [2/p - 3/2]$.
 - (c) a_R is $C^{k(p)+1}$ with

$$\begin{aligned} \|a_R\|_\infty &\leq c_R/|R|^{1/2}, \\ \|\partial^m a_R / \partial x_j^m\|_\infty &\leq c_R/|I_j|^m |R|^{1/2}, \quad j = 1, 2, \\ m &\leq k(p) + 1, \end{aligned}$$

and $\sum_R c_R^2 \leq A|\Omega|^{1-2/p}$, where A is an absolute constant.

With this definition of p -atom, Chang and Fefferman have shown the following result (see [1], [2]).

Theorem 1.5. *Let $f \in H^p(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$, $0 < p \leq 1$. Then (identifying f with its boundary values) f can be written as $f = \sum \lambda_k a_k$ where a_k are p -atoms and $\lambda_k \geq 0$ satisfy*

$$\sum \lambda_k^p \leq A_p \|f\|_{H^p}^p.$$

2. FACTORIZATION IN HARDY SPACE OF BI-UPPER HALF PLANE ($0 < p \leq 1$)

The main purpose of this paper is to study factorization theorems in Hardy spaces H^p , $0 < p \leq 1$, on the bi-upper half plane. The case of the bidisc can then be easily obtained by use of the Cayley transform.

Theorem 2.1 (Factorization Theorem for $\mathbf{R}_+^2 \times \mathbf{R}_+^2$). *Let $0 < p \leq 1$ and $f \in H^p(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$. Then there exist $g_j, h_j \in H^{2p}(\mathbf{R}_+^2 \times \mathbf{R}_+^2)$, such that*

$$f = \sum_{j=1}^{\infty} g_j h_j$$

in the sense of distributions, and $\sum \|g_j\|_{H^{2p}} \|h_j\|_{H^{2p}} \approx \|f\|_{H^p}$.

Similarly to the idea given by Coifman, Rochberg and Weiss [4], with the aid of Theorem 1.5, the proof of Theorem 2.1 can be reduced to the following theorem, that is, it suffices to prove Theorem 2.1 for $f = S(a)$, where a is a p -atom.

Theorem 2.2. *Let a be a p -atom, $0 < p \leq 1$. There exist $B_j, C_j \in H^{2p}(\mathbf{R}^2)$ such that (with $A = S(a)$)*

$$A = \sum_1^{\infty} B_j C_j$$

(in the sense of distributions) and

$$\sum_1^{\infty} \|B_j\|_{H^{2p}} \|C_j\|_{H^{2p}} \approx \|A\|_{H^p} \approx 1.$$

By Definition 1.4 a can be written as follows: $a = \sum_{R \in \mathcal{R}_a} a_R$, where a_R is supported in a rectangle $R \subseteq \Omega$ and the R in the sum have the property that no one R is contained in the triple of any other. Moreover,

$$\|a_R\|_{\infty} \leq c_R |R|^{-1/2}, \quad \sum_R c_R^2 \leq A |\Omega|^{1-2/p},$$

where A is an absolute constant.

Therefore, the proof of Theorem 2.2 is reduced to Theorem 2.3, as we shall show below.

Theorem 2.3. *Let a be a p -atom, $0 < p \leq 1$. Then for each $R \in \mathcal{R}_a$, there exist $B_j, C_j \in H^{2p}(\mathbf{R}^2)$ such that*

$$A_R = \sum_1^4 B_j C_j, \quad \text{where } A_R = S(a_R)$$

and

$$\sum_1^4 \|B_j\|_{H^{2p}} \|C_j\|_{H^{2p}} \leq c c_R |R|^{1/p-1/2}.$$

Proof. By shift, without loss of generality, we may assume that $R = l_1 I \times l_2 I$, where $I = [-1, 1]$, $0 < l_j < \infty$ (in general, $0 < l_j \ll 1$), $j = 1, 2$. Since

$$\begin{aligned} 1 &= \int_R \frac{1}{|R|} dw \\ &= \int_{l_1 I} \frac{1}{2l_1} \int_{l_2 I} \frac{1}{2l_2} dw_2 dw_1 \\ &= \prod_{j=1}^2 \int_{l_j I} \frac{1}{2l_j} \frac{1}{(z_j - w_j)} (z_j - w_j) dw_j \\ &= g_1(z) + g_2(z) - g_3(z) - g_4(z), \end{aligned}$$

where

$$\begin{aligned} g_1(z) &= z_1 z_2 \int_{l_1 I} \frac{1}{4l_1 l_2} \int_{l_2 I} \frac{dw_2 dw_1}{(z_1 - w_1)(z_2 - w_2)}, \\ g_2(z) &= \int_{l_1 I} \int_{l_2 I} \frac{w_1 w_2}{4l_1 l_2} \frac{dw_2 dw_1}{(z_1 - w_1)(z_2 - w_2)}, \\ g_3(z) &= \int_{l_1 I} \int_{l_2 I} \frac{w_1 z_2}{4l_1 l_2} \frac{dw_2 dw_1}{(z_1 - w_1)(z_2 - w_2)}, \\ g_4(z) &= \int_{l_1 I} \int_{l_2 I} \frac{z_1 w_2}{4l_1 l_2} \frac{dw_2 dw_1}{(z_1 - w_1)(z_2 - w_2)}. \end{aligned}$$

Thus

$$A_R(z) = A_R(z)g_1(z) + A_R(z)g_2(z) - A_R(z)g_3(z) - A_R(z)g_4(z).$$

We shall work on each $A_R(z)g_j(z)$ to achieve the desired factorization as claimed in Theorem 2.3.

Step 1. We shall work on $A_R(z)g_1(z)$.

Let η be a number determined by

$$-1 + \frac{1}{2p} < \eta < -1 + \frac{1}{2p} + \varepsilon, \quad \text{with } 0 < \varepsilon \ll 1.$$

Let

$$\begin{aligned} B_1(z) &= z_1^{-\eta} z_2^{-\eta} \int_R \frac{dw_1 dw_2}{(z_1 - w_1)(z_2 - w_2)}; \\ C_1(z) &= z_1^{1+\eta} z_2^{1+\eta} A_R(z) |R|^{-1}. \end{aligned}$$

Then $g_1(z)A_R(z) = B_1(z)C_1(z)$.

We shall show first

$$(2) \quad \|B_1\|_{H^{2p}} \|C_1\|_{H^{2p}} \leq c c_R |R|^{1/p-1/2}.$$

Lemma 2.4. $\int_{\mathbf{R}^2} |B_1(z)|^{2p} dz \leq c_p |R|^{1-2p\eta}.$

Proof. *Case 1.* $|z_1| > 2l_1, |z_2| > 2l_2.$

Then (see the proof of Lemma 1.2)

$$|B_1(z)| \leq c |z_1|^{-(\eta+1)} |z_2|^{-(\eta+1)} |R|.$$

Thus, since $2p(1+\eta) > 1$

$$\begin{aligned} \int_{(\mathbf{R} \setminus 2l_1 I) \times (\mathbf{R} \setminus 2l_2 I)} |B_1(z)|^{2p} dz_1 dz_2 &\leq c^{2p} |R|^{2p} \int_{2l_1}^{\infty} \int_{2l_2}^{\infty} t^{-2p(1+\eta)} s^{-2p(1+\eta)} ds dt \\ &\leq c |R|^{2p} l_1^{-2p(1+\eta)+1} l_2^{-2p(1+\eta)+1} \\ &= c |R|^{1-2p\eta}. \end{aligned}$$

Case 2. $|z|_1 \leq 2l_1, |z|_2 \leq 2l_2$.

Let $q = (1+\eta_0)/\eta_0$ where $\eta_0 > 0$ is chosen so that $\eta < \eta_0$ and $2p\frac{\eta}{\eta_0}(1+\eta_0) < 1$. Then $q' := q/(q-1) = 1+\eta_0$. Since $\frac{\eta}{\eta_0}2p(1+\eta_0) < 1$ if and only if $2p\eta(1+\eta_0) < \eta_0$ if and only if $2p\eta < \eta_0(1-2p\eta)$, and since we know that $0 < 1-2p\eta < 2p$, we have $\eta_0(1-2p\eta) < 2p\eta_0$. So we may choose such η_0 . By Hölder's inequality with exponent q , and by Lemma 1.1,

$$\begin{aligned} &\int_{2l_1 I \times 2l_2 I} |B_1(z)|^{2p} dz_1 dz_2 \\ &= \int_{2R} |z_1 z_2|^{-2p\eta} \left| \int_R \frac{dw_1 dw_2}{(z_1 - w_1)(z_2 - w_2)} \right|^{2p} dz_1 dz_2 \\ &\leq \left(\int_{2R} |z_1 z_2|^{-2p\eta \cdot \frac{(1+\eta_0)}{(1+\eta_0)-1}} dz \right)^{\frac{(1+\eta_0)-1}{(1+\eta_0)}} \\ &\quad \times \left(\int_{2R} \left| \int_R \frac{dw}{(z_1 - w_1)(z_2 - w_2)} \right|^{2p(1+\eta_0)} dz \right)^{\frac{1}{1+\eta_0}} \\ &\leq \left(\int_{2R} |z_1 z_2|^{-\frac{2p\eta(1+\eta_0)}{\eta_0}} dz \right)^{\frac{\eta_0}{1+\eta_0}} \left(\int_{2R} 1 dw \right)^{\frac{1}{1+\eta_0}}. \end{aligned}$$

Thus

$$\begin{aligned} \int_{2R} |B_1(z)|^{2p} dz_1 dz_2 &\leq c \left(|R|^{-\frac{2p\eta(1+\eta_0)}{\eta_0}+1} \right)^{\frac{\eta_0}{1+\eta_0}} |R|^{\frac{1}{1+\eta_0}} \\ &= c |R|^{\frac{-2p\eta(1+\eta_0)+\eta_0+1}{1+\eta_0}} = c |R|^{-2p\eta+1}. \end{aligned}$$

Case 3. $|z|_1 \leq 2l_1$ and $|z|_2 > 2l_2$.

Using part 2 of Lemma 1.2 and Lemma 1.3, we have

$$\begin{aligned} &\int_{2l_1 I \times (\mathbf{R} \setminus 2l_2 I)} |B_1(z)|^{2p} dz_1 dz_2 \\ &= \int_{2l_1 I \times (\mathbf{R} \setminus 2l_2 I)} |z_1 z_2|^{-2p\eta} \left| \int_R \frac{dw}{(z_1 - w_1)(z_2 - w_2)} \right|^{2p} dz \\ &\leq \int_{\mathbf{R} \setminus 2l_2 I} |z_2|^{-2p\eta-2p} l_2^{2p} dz_2 \int_{2l_1 I} |z_1|^{-2p\eta} \left| \int_R \frac{dw_1}{(z_1 - w_1)} \right|^{2p} dz_1 \\ &\leq c_p l_2^{2p} l_2^{-2p(1+\eta)+1} \int_{2l_1 I} |z_1|^{-2p\eta} \left| \int_{l_1 I} \frac{dw_1}{(z_1 - w_1)} \right|^{2p} dz_1. \end{aligned}$$

By using the argument of Case 2 (but in one variable only), we have

$$\int_{2l_1 I} |z_1|^{-2p\eta} \left| \int_{l_1 I} \frac{dw_1}{(z_1 - w_1)} \right|^{2p} dz_1 \leq c l_1^{-2p\eta+1}.$$

Thus

$$\int_{2l_1 I \times (\mathbf{R} \setminus 2l_2 I)} |B_1(z)|^{2p} dz \leq c_p l_2^{-2p\eta+1} l_1^{-2p\eta+1} \leq c_p |R|^{-2p\eta+1}.$$

Case 4. $|z_1| > 2l_1$ and $|z_2| \leq 2l_2$. By a similar argument to Case 3, we have

$$\int_{(\mathbf{R} \setminus 2l_1 I) \times 2l_2 I} |B_1(z)|^{2p} dz \leq c_p |R|^{-2p\eta+1}.$$

Therefore

$$\int_{\mathbf{R}^2} |B_1(z)|^{2p} dz \leq c_p |R|^{-2p\eta+1}.$$

Next we shall prove that

$$\int_{\mathbf{R}^2} |C_1(z)|^{2p} dz \leq c_p c_R^{2p} |R|^{2-p} |R|^{2p\eta-1} = c_p c_R^{2p} |R|^{1+2p\eta-p},$$

and this will complete the proof of (2). We state this as another lemma.

Lemma 2.5.

$$\int_{\mathbf{R}^2} |C_1(z)|^{2p} dz \leq c_p c_R^{2p} |R|^{1+2p\eta-p}.$$

Proof. Case 1. $z_1, z_2 \in 2R$.

Then

$$\begin{aligned} \int_{2R} |C_1(z)|^{2p} dz &= \int_{2R} |z_1 z_2|^{2p(1+\eta)} |A_R(z)|^{2p} dz_1 dz_2 |R|^{-2p} \\ &\leq c_p |R|^{2p(1+\eta)-2p} \int_{2R} |A_R(z)|^{2p} dz \\ &\leq c_p |R|^{2p\eta} \|a_R\|_2^{2p} |R|^{1-p} \\ &\leq c_p |R|^{2p\eta} c_R^{2p} |R|^{1-p} \\ &= c_p c_R^{2p} |R|^{2p\eta+1-p}. \end{aligned}$$

Case 2. $|z_1| > 2l_1, |z_2| > 2l_2$.

To prove this case, we need the following sublemmas. First let us introduce some notation. For $z_j, w_j \in \mathbf{R}$ and $j = 1, 2$, let

$$S_j^* = S_j^*(z_j, w_j) := \sum_{m=0}^{k(p)} \frac{1}{m!} \frac{\partial^m S_j(z_j, 0)}{\partial w_j^m} w_j^m.$$

Note that $S_j^*(z_j, w_j)$ is the $k(p)$ th partial sum of the Taylor expansion of $S_j^*(z_j, \cdot)$ at 0.

Sublemma 2.6. For a function ϕ , let

$$\phi^*(z_j) = \int_{l_j I} [S_j(z_j, w_j) - S_j^*(z_j, w_j)] \phi(w_j) dw_j.$$

Then for $|z_j| > 2l_j$, we have

$$|\phi^*(z_1, w_2)| \leq cl_1^{k(p)+1} |z_1|^{-(k(p)+2)} \int_{l_1 I} |\phi(w_1)| dw_1.$$

In particular, if

$$M(z_2, w_1) = \int_{l_2 I} [S_2(z_2, w_2) - S_2^*(z_2, w_2)] a_R(w) dw_2,$$

then

$$|M(z_2, w_1)| \leq cl_2^{k(p)+1} |z_2|^{-(k(p)+2)} \int_{l_2 I} |a_R(w)| dw_2.$$

Proof. Since $\forall m \in \mathbb{Z}^+$,

$$\frac{\partial^m S_j(z_j, w_j)}{\partial w_j^m} = c(-1)^m m! (z_j - w_j)^{-(m+1)},$$

we have

$$S_j(z_j, w_j) - S_j^* = \frac{\partial^{k(p)+1} S_j(z_j, w'_j)}{\partial w_j^{k(p)+1}} z_j^{-(k(p)+2)} w_j^{-(k(p)+1)}$$

where w'_j lies between 0 and w_j .

Then

$$|S_j(z_j, w_j) - S_j^*| \leq c |z_j - w'_j|^{-(k(p)+2)} |w_j|^{-(k(p)+1)}.$$

Claim. $|S_j(z_j, w_j) - S_j^*| \leq c |z_j|^{-(k(p)+2)} |w_j|^{-(k(p)+1)}$.

To prove this we need: $|z_j - w'_j|^{-(k(p)+2)} \leq c |z_j|^{-(k(p)+2)}$, that is, we have to show $|z_j - w'_j|^{-1} \leq c |z_j|^{-1}$.

Since

$$\begin{aligned} |z_j| &= |z_j - w'_j + w'_j| \\ &\leq |z_j - w'_j| + |w'_j| \\ &\leq |z_j - w'_j| + l_j \quad (\text{because } |w'_j| \leq |w_j| \leq l_j), \end{aligned}$$

we obtain

$$\begin{aligned} |z_j - w'_j| &\geq |z_j| - l_j \\ &\geq |z_j| - \frac{1}{2} |z_j| \quad (\text{because } |z'_j| > 2l_j) \\ &= \frac{1}{2} |z_j|. \end{aligned}$$

Hence $|z_j - w'_j|^{-1} \leq 2 |z_j|^{-1}$. Therefore the claim is proved.

Thus

$$\begin{aligned} |\phi^*(z_j)| &\leq c \int_{l_j I} |\phi(w_j)| |z_j|^{-(k(p)+2)} |w_j|^{k(p)+1} dw_j \\ &\leq cl_j^{k(p)+1} |z_j|^{-(k(p)+2)} \int_{l_j I} |\phi(w_j)| dw_j. \end{aligned}$$

Sublemma 2.7. If $|z_1| > 2l_1$ and $|z_2| > 2l_2$, then

$$|A_R(z)| \leq cc_R |R|^{k(p)+3/2} |z_1 z_2|^{-(k(p)+2)}.$$

Proof. By property 2(b) of Definition 1.4,

$$\begin{aligned} A_R(z) &= \int_{l_1 I} S_1(z_1, w_1) \int_{l_2 I} S_2(z_2, w_2) a_R(w) dw_2 dw_1 \\ &= \int_{l_1 I} [S_1(z_1, w_1) - S_1^*] \int_{l_2 I} [S_2(z_2, w_2) - S_2^*] a_R(w) dw_2 dw_1 \\ &= \int_{l_1 I} [S_1(z_1, w_1) - S_1^*] M(z_2, w_1) dw_1. \end{aligned}$$

Sublemma 2.6 implies

$$\begin{aligned} |A_R(z)| &\leq c l_1^{k(p)+1} |z_1|^{-(k(p)+2)} \int_{l_1 I} |M(z_2, w_1)| dw_1 \\ &\leq c l_1^{k(p)+1} |z_1|^{-(k(p)+2)} l_2^{k(p)+1} |z_2|^{-(k(p)+2)} \int_{l_1 I} |a_R(w_1, w_2)| dw_1 dw_2 \\ &\leq c |R|^{k(p)+3} |z_1 z_2|^{-(k(p)+2)}. \end{aligned}$$

Now we return to the proof of Case 2.

$$\begin{aligned} &\int_{(\mathbf{R} \setminus 2l_1 I) \times (\mathbf{R} \setminus 2l_2 I)} |C_1(z)|^{2p} dz \\ &= \int_{(\mathbf{R} \setminus 2l_1 I) \times (\mathbf{R} \setminus 2l_2 I)} |R|^{-2p} |z_1 z_2|^{-2p} |z_1 z_2|^{2p(1+\eta)} |A_R(z)|^{2p} dz_1 dz_2 \\ &\leq c c_R^{2p} \int_{(\mathbf{R} \setminus 2l_1 I) \times (\mathbf{R} \setminus 2l_2 I)} |R|^{2p(k(p)+1/2)} |z_1 z_2|^{2p(1+\eta)-2p(k(p)+2)} dz_1 dz_2 \\ &= c c_R^{2p} |R|^{2p(k(p)+1/2)} \int_{(\mathbf{R} \setminus 2l_1 I) \times (\mathbf{R} \setminus 2l_2 I)} |z_1 z_2|^{-2p(k(p)+1-\eta)} dz_1 dz_2. \end{aligned}$$

Note that

$$\begin{aligned} 2p(k(p) + 1 - \eta) &= 2pk(p) + 2p - 2p\eta \\ &\geq 2p(2/p - 3/2 - 1) + 2p - 2p\eta \\ &= 4 - 3p + 2p - 2p\eta \\ &> 4 - p - (-2p + 1 + \varepsilon) \\ &= 4 + p - 1 - \varepsilon \\ &= 3 + p - \varepsilon > 1. \end{aligned}$$

Thus, by Lemma 1.3,

$$\int_{\mathbf{R} \setminus 2l_1 I} \int_{\mathbf{R} \setminus 2l_2 I} |z_1 z_2|^{-2p(k(p)+1-\eta)} dz_1 dz_2 \leq c(l_1 l_2)^{-2p(k(p)+1-\eta)+1}.$$

Hence

$$\begin{aligned} \int_{(\mathbf{R} \setminus 2l_1 I) \times (\mathbf{R} \setminus 2l_2 I)} |C_1(z)|^{2p} dz &\leq c_p c_R^{2p} |R|^{2p(k(p)+1/2)-2p(k(p)+1-\eta)+1} \\ &= c_p c_R^{2p} |R|^{-p+2p\eta+1}. \end{aligned}$$

Case 3. $|z_1| < 2l_1, |z_2| \leq 2l_2$.

The key to this case is the following sublemma.

Sublemma 2.8. *If $z_1 \in \mathbf{R} \setminus 2l_1I$, then*

$$\int_{2l_2I} |A_R|^{2p} dz_2 \leq c_p c_R^{2p} l_1^{2p(k(p)+3/2)} |z_1|^{-2p(k(p)+2)} l_2^{1-p}.$$

Proof.

$$\begin{aligned} & \int_{2l_2I} \left| \int_R S_1(z_1, w_1) S_2(z_2, w_2) a_R(w) dw \right|^{2p} dz_2 \\ &= \int_{2l_2I} \left| \int_{l_1I} [S_1(z_1, w_1) - S_1^*(z_1)] \int_{l_2I} S_2(z_2, w_2) a_R(w) dw \right|^{2p} dz_2 \\ &\leq \int_{2l_2I} \left[c l_1^{k(p)+3/2} |z_1|^{-(k(p)+2)} \int_{l_1I} \left| \int_{l_2I} S_2(z_2, w_2) a(w) dw \right|^{2p} dw_1 \right]^{2p} dz_2 \\ &\leq c_p l_1^{2p(k(p)+3/2)} |z_1|^{-2p(k(p)+2)} \left(\int_R |S_2(a_R)(w_1, z_2)|^2 dz_2 \right)^p \left(\int_{l_2I} dz_2 \right)^{1-p} \\ &\leq c_p l_1^{2p(k(p)+3/2)} |z_1|^{-2p(k(p)+2)} \left(\int_R |a_R(w_1, w_2)|^2 dw \right)^p |R|^{1-p} \\ &\leq c_p l_1^{2p(k(p)+3/2)} |z_1|^{-2p(k(p)+2)} c_R^{2p} l_2^{1-p} \\ &= c_p c_R^{2p} l_1^{2p(k(p)+3/2)} |z_1|^{-2p(k(p)+2)} l_2^{1-p}. \end{aligned}$$

With the aid of this sublemma we have

$$\begin{aligned} & \int_{(\mathbf{R} \setminus 2l_1) \times 2l_2I} |C_1(z)|^{2p} dz \\ &= \int_{\mathbf{R} \setminus 2l_1} \int_{2l_2I} |R|^{-2p} |z_1 z_2|^{2p(1+\eta)} |A_R(z)|^{2p} dz_2 dz_1 \\ &\leq \int_{\mathbf{R} \setminus 2l_1} |R|^{-2p} c_p c_R^{2p} l_1^{2p(k(p)+3/2)} (2l_2)^{2p(1+\eta)} l_2^{1-p} |z_1|^{-2p(k(p)+2)+2p(1+\eta)} dz_1 \\ &\leq c_p c_R^{2p} \int_{2l_1}^{\infty} l_1^{2p(k(p)+3/2)} l_2^{2p\eta+1+p} t^{-2p(k(p)+2)+2p(1+\eta)} dt |R|^{-2p} \\ &\leq c_p c_R^{2p} l_1^{2p(k(p)+3/2)-2p(k(p)+\eta+1)+1} l_2^{2p\eta+1+p} |R|^{-2p} \\ &= c_p c_R^{2p} l_1^{p+2p\eta+1} l_2^{2p\eta+1+p} |R|^{-2p} \\ &= c_p c_R^{2p} |R|^{-p+2p\eta+1}. \end{aligned}$$

Case 4. $|z_1| \leq 2l_1, |z_2| > 2l_2$.

By symmetry with Case 3, we obtain

$$\int_{\mathbf{R}^2} |C_1(z)|^{2p} dz \leq c_p c_R^{2p} |R|^{-p+2p\eta+1},$$

and hence the proof of (2) is complete.

Step 2. We now work on $g_2(z)A_R(z)$.

Put

$$g_2(z)A_R(z) = B_2(z)C_2(z),$$

where

$$B_2(z) = (z_1 z_2)^{-\eta} g_2(z) \quad \text{and} \quad C_2(z) = (z_1 z_2)^\eta A_R(z).$$

By using an argument similar to that of (2), we obtain

$$(3) \quad \|B_2\|_{H^{2p}} \|C_2\|_{H^{2p}} \leq c c_R |R|^{1/p-1/2}.$$

Similarly, if we set

$$g_3(z) A_R(z) = B_3(z) C_3(z),$$

where

$$B_3(z) = z_1^{-\eta} z_2^{-\eta-1} g_3(z) \quad \text{and} \quad C_3(z) = z_1^\eta z_2^{\eta+1} A_R(z)$$

and

$$g_4(z) A_R(z) = B_4(z) C_4(z),$$

where

$$B_4(z) = z_1^{-\eta-1} z_2^{-\eta-1} g_4(z) \quad \text{and} \quad C_4(z) = z_1^{\eta+1} z_2^{-\eta} A_R(z),$$

then we can prove

$$(4) \quad \|B_j\|_{H^{2p}} \|C_j\|_{H^{2p}} \leq c c_R |R|^{1/p-1/2}, \quad j = 3, 4.$$

Combining (2), (3), and (4) completes the proof of Theorem 2.3.

We now prove Theorem 2.2:

$$A = \sum_{R \subset \Omega} S(a_R) = \sum_{R \subset \Omega} \sum_{j=i}^4 B_j C_j$$

and because $2/p - 1 > 1$,

$$\begin{aligned} \sum_{R \subset \Omega} \sum_{j=i}^4 \|B_j\|_{H^{2p}} \|C_j\|_{H^{2p}} &\leq \sum_{R \subset \Omega} c_R |R|^{1/p-1/2} \\ &\leq \left(\sum_{R \subset \Omega} c_R^2 \right)^{1/2} \left(\sum_{R \subset \Omega} |R|^{2/p-1} \right)^{1/2} \\ &\leq \left(\sum_{R \subset \Omega} c_R^2 \right)^{1/2} \left(\sum_{R \subset \Omega} |R| \right)^{(2/p-1)/2} \\ &\leq ((A|\Omega|)^{1-2/p})^{1/2} ((2|\Omega|)^{2/p-1})^{1/2} = c. \end{aligned}$$

Therefore the proof of Theorem 2.2 is complete.

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REFERENCES

1. S.-Y. A. Chang and R. Fefferman *A continuous version of duality of H^1 with BMO on the bidisc*, Ann. of Math. (2) **112** (1980), 179–201. MR **82a**:32009
2. ——— *The Calderon-Zygmund decomposition on product domains*, Amer. J. Math. **104** (1982), 455–468. MR **84a**:42028
3. R. Coifman *A real variable characterization of H^p* , Studia Math. **51** (1974), 269–274. MR **50**:10784
4. R. Coifman, R. Rochberg, and G. Weiss *Factorization theorem for Hardy spaces in several variables*, Ann. of Math. (2) **103** (1976), 611–635. MR **54**:843
5. S. G. Krantz *Function theory of several complex variables*, Wadsworth and Brooks/Cole, Belmont, CA, 1992. MR **93c**:32001
6. S. G. Krantz and Song-Ying Li *On the decompositions for Hardy spaces in domains in C^n and applications*, preprint, 1992.
7. R. Latter *A characterization of $H(\mathbf{R}^n)^p$ in terms of atoms*, Studia Math. **62** (1978), 93–101. MR **58**:2198
8. E. M. Stein *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, NJ, 1970. MR **44**:7280

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