WEIGHTED INEQUALITIES FOR SOME ONE-SIDED OPERATORS

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(Communicated by J. Marshall Ash)

ABSTRACT. We give a characterization of the pairs of weights (u, v) such that the Weyl fractional integral operator maps $L^p(vdx)$ into weak $L^q(udx)$, $1 < p \leq q < \infty$ or $p = 1 < q < \infty$. For the case $p < q$ we give necessary and sufficient conditions for the weak type of a maximal operator that includes as particular cases the Weyl fractional integral, the dual of the Hardy operator and the fractional one-sided maximal operator. As a consequence we give a new characterization of the pairs of weights for which the fractional one-sided maximal operator is bounded.

1. INTRODUCTION

Let f be a locally integrable function on $\mathbb{R}$ and $0 < \alpha < 1$. The Weyl fractional integral is defined as

$$(W_\alpha f)(x) = \int_x^\infty \frac{f(y)}{(y-x)^{1-\alpha}} dy.$$

It is known that $p > 1$, $1/p > \alpha$ and $1/q = 1/p - \alpha$ imply W_α maps L^p into L^q .

In 1988 Andersen and Sawyer [AS] characterized those u for which it is true that

$$\left(\int_{\mathbb{R}} |W_\alpha f|^q u^q \right)^{1/q} \leq C \left(\int_{\mathbb{R}} |f|^p u^p \right)^{1/p}$$

with the above restrictions on p, q, α . In this note we study the weak type inequalities for this operator in the case $p = q$ and in the case $p < q$ we obtain necessary and sufficient conditions for the weak type of the operator

$$N^+ f(x) = \sup_{x < c} h(x, c) \int_x^c f(y) k(x, y) dy,$$

where $h(s, t)$ and $k(s, t)$ are positive measurable functions defined on $s \leq t$. This operator includes as particular cases the Weyl fractional integral ($h \equiv 1$ and $k(s, t) = (t-s)^{\alpha-1}$), the fractional one-sided Hardy-Littlewood maximal operator ($k \equiv 1$ and $h(s, t) = (t-s)^{-\alpha}$) and the dual of the Hardy operator ($h \equiv 1$

Received by the editors March 15, 1994 and, in revised form, September 14, 1994.

1991 *Mathematics Subject Classification*. Primary 26A33.

Key words and phrases. Weyl fractional integral, weights.

This research has been supported by D.G.I.C.Y.T. grant (PB91-0413) and Junta de Andalucía.

and $k \equiv 1$). Finally we apply this result together with those in [MT1] to obtain a new characterization of the pairs of weights for which the fractional one-sided Hardy-Littlewood maximal operator is bounded. By weights we understand locally integrable positive functions.

Throughout the paper, C will be a constant that may change from line to line. If A is any measurable set and g a positive function, $g(A)$ will stand for the integral of g over A and if $1 < p < \infty$, then p' will denote its conjugate exponent. M_u will denote the maximal operator defined by

$$M_u g(x) = \sup_{x \in I} \frac{1}{\int_I u} \int_I |gu|,$$

which is known to be of weak type $(1, 1)$ with respect to the measure udx . Our main results are the following theorems.

Theorem 1. *Let $1 \leq p < q < \infty$ and consider the following two conditions:*

(1) *There exists C such that*

$$(1.1) \quad u(\{x : N^+ f(x) > \lambda\}) \leq C \left(\frac{1}{\lambda^p} \int f^p v \right)^{q/p}.$$

(2) *There exists C such that for any $a < b < c$*

$$(1.2) \quad h(a, c) \left(\int_a^b u \right)^{1/q} \left(\int_b^c v^{1-p'}(y) k^{p'}(a, y) dy \right)^{1/p'} \leq C, \quad p > 1,$$

or there exists C such that for any $a < b$ and almost all $c > b$

$$h(a, c) \left(\int_a^b u \right)^{1/q} \leq C v(c) k^{-1}(a, c), \quad p = 1.$$

If h is non-increasing on its second variable, then $(1.2) \implies (1.1)$. If h and k are non-decreasing on the first variable, then $(1.1) \implies (1.2)$.

Theorem 2. *Let $R_\alpha f(x) = \int_{-\infty}^x \frac{f(y)}{(x-y)^{1-\alpha}} dy$ be the formal adjoint of W_α . Let $1 < p \leq q < \infty$ or $p = 1 < q < \infty$. Then (1.1) for W_α holds iff: There exists C such that for any bounded interval I*

$$(1.3) \quad \int_I (R_\alpha(\chi_I u))^{p'} v^{1-p'} \leq C \left(\int_I u \right)^{p'/q'}, \quad \text{if } 1 < p \leq q < \infty,$$

or

$$\|R_\alpha(\chi_I u) v^{-1}\|_{L^\infty(v)} \leq C \left(\int_I u \right)^{1/q'}, \quad \text{if } p = 1 < q < \infty.$$

Theorem 3. Let $M_\alpha^+ f(x) = \sup_{h>0} h^{\alpha-1} \int_x^{x+h} |f|$. Then M_α^+ is bounded from $L^p(v)$ to $L^q(u)$, $1 < p < q < \infty$, if, and only if, there exists C such that for any $a < b$

$$\left(\int_{-\infty}^a \frac{u(y)}{(b-y)^{(1-\alpha)q}} dy \right)^{\frac{1}{q}} \left(\int_a^b v^{1-p'} \right)^{\frac{1}{p'}} \leq C.$$

Remarks. (1) One can, of course, change the orientation of the real line and obtain theorems for R_α and for the operator

$$N^- f(x) = \sup_{c < x} h(c, x) \int_c^x f(y) k(y, x) dy.$$

(2) Sufficient conditions for the weak type of the Weyl fractional integral were obtained by Kokilashvili and Gabidzashvili in [GK].

(3) Theorem 2 for the two-sided fractional integral was obtained first by E. Sawyer [S].

2. PROOFS OF THE THEOREMS

Proof of Theorem 1. (1.1) $\implies$ (1.2). Assume first $p > 1$. We fix $a < b < c$, and consider the function

$$f(y) = v^{1-p'}(y) k^{(p'-1)}(a, y) \chi_{(b,c)}(y).$$

If $a < x < b$, we have

$$\begin{aligned} N^+ f(x) &\geq h(x, c) \int_b^c v^{1-p'}(y) k^{(p'-1)}(a, y) k(x, y) dy \\ &\geq h(a, c) \int_b^c v^{1-p'}(y) k^{p'}(a, y) dy = \lambda. \end{aligned}$$

This means that $(a, b) \subset \{x : N^+ f(x) > \lambda\}$ and then

$$\lambda^q \int_a^b u \leq C \left(\int_b^c v^{1-p'}(y) k^{p'}(a, y) dy \right)^{\frac{q}{p}}.$$

If $\lambda < \infty$ this is equivalent to (1.2). If $\lambda = \infty$, then $f(y) = v^{-1}(y) k(a, y) \notin L^{p'}(v \chi_{(b,c)})$ and therefore there exists a non-negative function $g \in L^p(v)$ such that $\infty = \int f g v = \int_b^c g(y) k(a, y) dy$. But then for any $x \in (a, b)$

$$N^+ g(x) \geq h(a, c) \int_b^c g(y) k(x, y) dy \geq h(a, c) \int_b^c g(y) k(a, y) dy = \infty,$$

which contradicts (1.1).

The case $p = 1$ is obtained taking $a < b < c - t < c$, considering $f = \chi_{(c-t, c)}$ and observing that then $x \in (a, b) \implies N^+ f(x) \geq h(x, c) \int_{c-t}^c k(x, y) dy \geq h(a, c) \int_{c-t}^c k(a, y) dy = \lambda$ and therefore

$$h(a, c) \int_{c-t}^c k(a, y) dy \left(\int_a^b u \right)^{1/q} \leq C \int_{c-t}^c v,$$

which implies $h(a, c)(u(a, b))^{1/q} \leq Cv(c)k^{-1}(a, c)$ by Lebesgue's Differentiation Theorem.

Let us assume now that (1.2) holds and that h is non-increasing on the second variable. We fix $\lambda > 0$ and $f \geq 0$ and x such that $N^+f(x) > \lambda$. Then there exists $c > x$ such that $h(x, c) \int_x^c f(y)k(x, y) dy > \lambda$. Assume $p > 1$. For any $x < b < c$ we may write

$$\begin{aligned} \lambda &< h(x, c) \int_x^b f(y)k(x, y) dy + h(x, c) \int_b^c f(y)k(x, y) dy \\ &\leq h(x, b) \int_x^b f(y)k(x, y) dy + h(x, c) \left(\int_b^c f^p v \right)^{\frac{1}{p}} \left(\int_b^c v^{1-p'}(y)k^{p'}(x, y) dy \right)^{\frac{1}{p'}} \\ &\leq h(x, b) \int_x^b f(y)k(x, y) dy + C \left(\int_b^c f^p v \right)^{\frac{1}{p}} \left(\int_x^b u \right)^{-\frac{1}{q}}. \end{aligned}$$

By continuity of the integral we can choose b so that

$$C \left(\int_b^c f^p v \right)^{\frac{1}{p}} \left(\int_x^b u \right)^{-\frac{1}{q}} = \lambda/2,$$

which implies $\lambda/2 < h(x, b) \int_x^b f(y)k(x, y) dy$.

We define a sequence of points in (x, b) as follows: $x_0 = b$, $u(x, x_{k+1}) = \frac{1}{2}u(x, x_k)$. From the definition it follows easily that $x_0 = b > x_1 > \dots > x_k > \dots > x$, $\lim x_k = x$. We may therefore write

$$\begin{aligned} \lambda/2 &< h(x, b) \int_x^b f(y)k(x, y) dy = h(x, b) \sum_{k=0}^{\infty} \int_{x_{k+1}}^{x_k} f(y)k(x, y) dy \\ &= h(x, b) \sum_{k=0}^{\infty} \int_{x_{k+1}}^{x_k} f(y)v^{1/p}(y)v^{-1/p}(y)k(x, y) dy. \end{aligned}$$

Hölder's inequality gives

$$\begin{aligned} \frac{\lambda}{2} &< h(x, b) \sum_{k=0}^{\infty} \left(\int_{x_{k+1}}^{x_k} f^p v \right)^{1/p} \left(\int_{x_{k+1}}^{x_k} v^{1-p'}(y)k^{p'}(x, y) dy \right)^{1/p'} \\ &\leq C \sum_{k=0}^{\infty} \frac{h(x, b)}{h(x, x_k)} \left(\int_{x_{k+1}}^{x_k} f^p v \right)^{1/p} \left(\int_x^{x_{k+1}} u \right)^{-1/q} \\ &\leq C \sum_{k=0}^{\infty} \left(\int_x^{x_{k+1}} u \right)^{1/p-1/q} \left(\frac{\int_x^{x_k} f^p v u^{-1} u}{\int_x^{x_k} u} \right)^{1/p} \\ &\leq C \sum_{k=0}^{\infty} \left(\int_x^{x_{k+1}} u \right)^{1/p-1/q} (M_u f^p v u^{-1})^{1/p}(x) \\ &\leq Cu(x, b)^{1/p-1/q} (M_u f^p v u^{-1})^{1/p}(x), \end{aligned}$$

where the last inequality follows from the facts that $1/p - 1/q > 0$ and $\int_x^{x_k} u = 2^{-k}u(x, b)$. Since $\int_x^b u = C \left(\int_b^c f^p v \right)^{\frac{q}{p}} \lambda^{-q}$, this last inequality gives

$$\lambda^q < C \left(\int_b^c f^p v \right)^{\frac{q}{p}-1} (M_u f^p v u^{-1})(x) \leq C \left(\int_{\mathbb{R}} f^p v \right)^{\frac{q}{p}-1} (M_u f^p v u^{-1})(x).$$

We have therefore proved that

$$\{x : N^+ f(x) > \lambda\} \subset \{x : (M_u f^p v u^{-1})(x) > C \lambda^q \left(\int_{\mathbb{R}} f^p v \right)^{1-\frac{q}{p}}\},$$

and (1.1) follows from the fact that the operator M_u is of weak type one-to-one with respect to the measure udx .

The case $p = 1$ is proved in a similar way.

Proof of Theorem 2. Let us assume that (1.1) for W_α holds. If $p > 1$ we have

$$\begin{aligned} \left(\int_I (R_\alpha(\chi_I u))^{p'} v^{1-p'} \right)^{1/p'} &\leq \|R_\alpha(\chi_I u) v^{-1}\|_{L^{p'}(v)} \\ &= \sup_{\{f \geq 0 : \|f\|_{L^p(v)}=1\}} \int v^{-1} R_\alpha(\chi_I u) f v \\ &= \sup_{\{f \geq 0 : \|f\|_{L^p(v)}=1\}} \int_I (W_\alpha f) u \\ &= \sup_{\{f \geq 0 : \|f\|_{L^p(v)}=1\}} \int_0^\infty u(\{x \in I : W_\alpha f(x) > \lambda\}) d\lambda \\ &\leq \int_0^\infty \min(u(I), C \lambda^{-q}) d\lambda = C u(I)^{1/q'}. \end{aligned}$$

The case $p = 1$ is treated in the same way.

To prove that the condition is sufficient we fix $f \geq 0$ and $\lambda > 0$. Let $O_\lambda = \{x : W_\alpha f(x) > \lambda\} = \bigcup I_k$ where $I_k = (a_k, b_k)$ are disjoint open intervals. We fix $0 < \beta < 1$ and call F the set of k 's such that $\frac{1}{u(I_k)} \int_{I_k} W_\alpha(f \chi_{I_k}) u > \beta \lambda$ and G all the other k 's.

If $k \in F$, then

$$\lambda^q u(I_k) < u(I_k)^{1-q} \beta^{-q} \left(\int_{I_k} W_\alpha(f \chi_{I_k}) u \right)^q = u(I_k)^{1-q} \beta^{-q} \left(\int_{I_k} f R_\alpha(u \chi_{I_k}) \right)^q.$$

If $p > 1$ we use Hölder's inequality and obtain

$$\begin{aligned} \lambda^q u(I_k) &\leq u(I_k)^{1-q} \beta^{-q} \left(\int_{I_k} f^p v \right)^{q/p} \left(\int_{I_k} (R_\alpha(u \chi_{I_k}))^{p'} v^{1-p'} \right)^{q/p'} \\ &\leq C u(I_k)^{1-q} \beta^{-q} \left(\int_{I_k} f^p v \right)^{q/p} u(I_k)^{q/q'} = C \beta^{-q} \left(\int_{I_k} f^p v \right)^{q/p}. \end{aligned}$$

If $p = 1$ we have

$$\begin{aligned}\lambda^q u(I_k) &\leq u(I_k)^{1-q} \beta^{-q} \left(\int_{I_k} f R_\alpha(u \chi_{I_k}) v v^{-1} \right)^q \\ &\leq C u(I_k)^{1-q} \beta^{-q} \left(\int_{I_k} f v \right)^q u(I_k)^{q/q'} = C \beta^{-q} \left(\int_{I_k} f v \right)^q.\end{aligned}$$

Let now $x \in (a_k, b_k)$ be such that $W_\alpha f(x) > 2\lambda$. Then

$$\begin{aligned}2\lambda &< \int_x^\infty \frac{f(y)}{(y-x)^{1-\alpha}} dy = \int_x^{b_k} \frac{f(y)}{(y-x)^{1-\alpha}} dy + \int_{b_k}^\infty \frac{f(y)}{(y-x)^{1-\alpha}} dy \\ &\leq (W_\alpha f \chi_{I_k})(x) + \int_{b_k}^\infty \frac{f(y)}{(y-b_k)^{1-\alpha}} dy \leq (W_\alpha f \chi_{I_k})(x) + \lambda.\end{aligned}$$

Therefore $\{x \in (a_k, b_k) : W_\alpha f(x) > 2\lambda\} \subset \{x \in I_k : W_\alpha(f \chi_{I_k})(x) > \lambda\}$.

Now if $k \in G$ we may write

$$\begin{aligned}u(\{x \in I_k : W_\alpha f(x) > 2\lambda\}) &\leq u(\{x \in I_k : W_\alpha(f \chi_{I_k})(x) > \lambda\}) \\ &= \int_{\{x \in I_k : W_\alpha(f \chi_{I_k})(x) > \lambda\}} u(x) dx \leq \frac{1}{\lambda} \int_{I_k} W_\alpha(f \chi_{I_k}) u \leq \beta u(I_k).\end{aligned}$$

We have thus proved that $\lambda^q u(I_k) \leq C \beta^{-q} \left(\int_{I_k} f^p v \right)^{q/p}$ if $k \in F$ and $u(\{x \in I_k : W_\alpha f(x) > 2\lambda\}) \leq \beta u(I_k)$ if $k \in G$. Therefore

$$\begin{aligned}(2\lambda)^q u(\{x : W_\alpha f(x) > 2\lambda\}) &= \sum_k (2\lambda)^q u(\{x \in I_k : W_\alpha f(x) > 2\lambda\}) \\ &\leq 2^q \sum_{k \in F} \lambda^q u(I_k) + 2^q \sum_{k \in G} \lambda^q u(\{x \in I_k : W_\alpha f(x) > 2\lambda\}) \\ &\leq C 2^q \beta^{-q} \sum_{k \in F} \left(\int_{I_k} f^p v \right)^{q/p} + 2^q \lambda^q \beta \sum_{k \in G} u(I_k) \\ &\leq C 2^q \beta^{-q} \left(\int f^p v \right)^{q/p} + 2^q \lambda^q \beta u(\{x : W_\alpha f(x) > \lambda\}).\end{aligned}$$

Choosing $\beta = \frac{1}{2^{q+1}}$ we have

$$(2\lambda)^q u(\{x : W_\alpha f(x) > 2\lambda\}) \leq C \left(\int f^p v \right)^{q/p} + \frac{1}{2} \lambda^q u(\{x : W_\alpha f(x) > \lambda\}).$$

For any positive t we have

$$\begin{aligned}A_t &= \sup_{0 < \lambda < t} \lambda^q u(\{x : W_\alpha f(x) > \lambda\}) \\ &\leq C \left(\int f^p v \right)^{q/p} + \frac{1}{2} \sup_{0 < \lambda < t/2} \lambda^q u(\{x : W_\alpha f(x) > \lambda\}) \\ &\leq C \left(\int f^p v \right)^{q/p} + \frac{A_t}{2},\end{aligned}$$

and it is enough to prove that A_t is finite. Let us consider the following condition:

there exists C such that for any $a < b$

$$(2.1) \quad \left(\int_a^b u \right)^{1/q} \left(\int_b^\infty \frac{v^{1-p'}(y)}{(y-a)^{(1-\alpha)p'}} dy \right)^{1/p'} \leq C, \quad p > 1,$$

or

$$\left(\int_a^b u \right)^{1/q} \leq C \operatorname{ess\,inf} \{v(y)(y-a)^{1-\alpha} : y > b\}, \quad p = 1.$$

Claim. (1.3) $\Rightarrow$ (2.1). If $p > 1$ we reason as follows: Let $a < b < c$ be such that $u(a, c) \leq 3u(a, b)$. Then

$$\begin{aligned} \int_b^c \frac{v^{1-p'}(y)}{(y-a)^{(1-\alpha)p'}} dy &= \int_b^c \left(\int_a^b u \right)^{p'} \frac{v^{1-p'}(y)}{(y-a)^{(1-\alpha)p'}} dy \left(\int_a^b u \right)^{-p'} \\ &\leq \int_b^c \left(\int_{-\infty}^y \frac{u(s)\chi_{(a,c)}(s)}{(y-s)^{(1-\alpha)}} ds \right)^{p'} v^{1-p'}(y) dy \left(\int_a^b u \right)^{-p'} \\ &\leq \int_a^c (R_\alpha u \chi_{(a,c)})^{p'} v^{1-p'} \left(\int_a^b u \right)^{-p'} \\ &\leq C \left(\int_a^c u \right)^{p'/q'} \left(\int_a^b u \right)^{-p'} \leq C \left(\int_a^b u \right)^{-p'/q}. \end{aligned}$$

We have thus seen that if $u(a, c) \leq 3u(a, b)$, then (1.3) implies

$$\left(\int_a^b u \right)^{1/q} \left(\int_b^c \frac{v^{1-p'}(y)}{(y-a)^{(1-\alpha)p'}} dy \right)^{1/p'} \leq C.$$

Let now $a < b < c$. Let us choose $x_0 = a$, $x_1 = b$, x_k such that $2^k u(a, b) = u(x_k, x_{k+1})$ and assume $x_N < c \leq x_{N+1}$. Then

$$\begin{aligned} &\left(\int_a^b u \right)^{p'/q} \int_b^c \frac{v^{1-p'}(y)}{(y-a)^{(1-\alpha)p'}} dy \\ &= \left(\int_a^b u \right)^{p'/q} \sum_{k=1}^{N-1} \int_{x_k}^{x_{k+1}} \frac{v^{1-p'}(y)}{(y-a)^{(1-\alpha)p'}} dy + \left(\int_a^b u \right)^{p'/q} \int_{x_N}^c \frac{v^{1-p'}(y)}{(y-a)^{(1-\alpha)p'}} dy \\ &\leq \sum_{k=1}^{N-1} (2^{-(k-1)})^{p'/q} \left(\int_{x_{k-1}}^{x_k} u \right)^{p'/q} \int_{x_k}^{x_{k+1}} \frac{v^{1-p'}(y)}{(y-x_{k-1})^{(1-\alpha)p'}} dy \\ &\quad + (2^{-(N-1)})^{p'/q} \left(\int_{x_{N-1}}^{x_N} u \right)^{p'/q} \int_{x_N}^c \frac{v^{1-p'}(y)}{(y-x_{N-1})^{(1-\alpha)p'}} dy \\ &\leq C \sum_{k=1}^N 2^{-(k-1)p'/q} \leq C. \end{aligned}$$

Letting c go to infinity we obtain (2.1). If $p = 1$ we fix $a < b$ and observe that for $y > b$ one has:

$$\begin{aligned} \frac{1}{v(y)(y-a)^{1-\alpha}} &= \int_a^b u(s) ds \frac{v^{-1}(y)}{(y-a)^{1-\alpha}} \left(\int_a^b u \right)^{-1} \\ &\leq \int_a^b \frac{u(s)\chi_{(a,b)}(s)}{(y-s)^{1-\alpha}} ds v^{-1}(y) \left(\int_a^b u \right)^{-1} \\ &\leq \int_{-\infty}^y \frac{u(s)\chi_{(a,b)}(s)}{(y-s)^{1-\alpha}} ds v^{-1}(y) \left(\int_a^b u \right)^{-1} \\ &= R_\alpha(u\chi_{(a,b)})(y) v^{-1}(y) \left(\int_a^b u \right)^{-1} \leq C \left(\int_a^b u \right)^{-1/q}; \end{aligned}$$

therefore,

$$\left(\int_a^b u \right)^{1/q} \leq C \operatorname{ess\,inf} \{v(y)(y-a)^{1-\alpha} : y > b\}.$$

Finally we will prove (2.1) implies that A_t is finite. It is enough to consider the case of small t , since

$$\begin{aligned} \sup_{t_1 < \lambda < t_2} \lambda^q u(\{x : W_\alpha f(x) > \lambda\}) &\leq t_2^q u(\{x : W_\alpha f(x) > t_1\}) \\ &\leq \left(\frac{t_2}{t_1} \right)^q \sup_{0 < \lambda < t_1 + \varepsilon} \lambda^q u(\{x : W_\alpha f(x) > \lambda\}). \end{aligned}$$

We may assume that f is bounded and with compact support. Let a and b be real numbers such that support of f is contained in (a, b) . Let us suppose then that t is so small that $\lambda < t \Rightarrow \left(\frac{1}{\lambda} \int_a^b f \right)^{\frac{1}{1-\alpha}} = s > b - a$. Then

$$\begin{aligned} \lambda^q u(\{x : W_\alpha f(x) > \lambda\}) &= \lambda^q u(\{x < a : W_\alpha f(x) > \lambda\}) + \lambda^q u(\{a < x < b : W_\alpha f(x) > \lambda\}) \\ &\leq \lambda^q u(\{x < a : W_\alpha f(x) > \lambda\}) + t^q u(a, b). \end{aligned}$$

We must then prove that

$$\sup_{0 < \lambda < t} \lambda^q u(\{x < a : W_\alpha f(x) > \lambda\}) < \infty.$$

But $x < a; W_\alpha f(x) > \lambda \Rightarrow \lambda < \int_a^b \frac{f(y)}{(y-x)^{1-\alpha}} dy \leq \frac{1}{(a-x)^{1-\alpha}} \int_a^b f \Rightarrow a-x < \left(\frac{1}{\lambda} \int_a^b f \right)^{\frac{1}{1-\alpha}} = s$, i.e. $\{x < a : W_\alpha f(x) > \lambda\} \subset (a-s, a)$, and thus

$$\lambda^q u(\{x < a : W_\alpha f > \lambda\}) \leq \lambda^q \int_{a-s}^a u = \left(\int_a^b f \right)^q s^{(\alpha-1)q} \int_{a-s}^a u.$$

If $p > 1$ we may write $\int_a^b f = \int_a^b f v^{1/p} v^{-1/p}$, use Hölder's inequality and get

$$\begin{aligned} \lambda^q u(\{x < a : W_\alpha f > \lambda\}) &\leq \left(\int_a^b f^p v \right)^{q/p} \int_{a-s}^a u \left(\int_a^b \frac{v^{1-p'}(y)}{s^{(1-\alpha)p'}} dy \right)^{q/p'} \\ &\leq C \left(\int_a^b f^p v \right)^{q/p} \int_{a-s}^a u \left(\int_a^b \frac{v^{1-p'}(y)}{(y-a+s)^{(1-\alpha)p'}} dy \right)^{q/p'} \leq C \left(\int_a^b f^p v \right)^{q/p}. \end{aligned}$$

We have used (2.1) and the fact that $s > b - a$ implies $y - a + s < 2s$. If $p = 1$ we have

$$\begin{aligned} \left(\int_a^b f \right)^q s^{(\alpha-1)q} \int_{a-s}^a u &\leq \left(\int_a^b f(y) \frac{v(y)v^{-1}(y)}{(y-a+s)^{(1-\alpha)}} dy \right)^q \int_{a-s}^a u \\ &\leq C \left(\int_a^b f v \right)^q. \end{aligned}$$

Proof of Theorem 3. The pairs of weights (u, v) for which the operator M_α^+ is bounded from $L^p(v)$ to $L^q(u)$ were characterized in [MT1] by the condition $S_{p,q,\alpha}^+$: $(\int_I (M_\alpha^+(\sigma\chi_I))^q u)^{\frac{1}{q}} \leq C (\int_I \sigma)^{\frac{1}{p}}$, where $\sigma = v^{1-p'}$. The following duality argument shows that this condition is implied by the weak type of the fractional integral R_α from $L^{q'}$ to $L^{p'}$ with respect to the weights $(u^{1-q'}, v^{1-p'})$. If we observe that $M_\alpha^+ f(x) \leq W_\alpha f(x)$ and define $B = \{g \geq 0 : \|g\|_{L^{q'}(u^{1-q'})} = 1\}$, we have:

$$\begin{aligned} \left(\int_I (M_\alpha^+(\sigma\chi_I))^q u \right)^{\frac{1}{q}} &= \left(\int_{\mathbb{R}} (M_\alpha^+(\sigma\chi_I) u^{\frac{1}{q}} \chi_I)^q \right)^{\frac{1}{q}} = \sup_B \int_{\mathbb{R}} M_\alpha^+(\sigma\chi_I) \chi_I g \\ &\leq \sup_B \int_{\mathbb{R}} W_\alpha(\sigma\chi_I) \chi_I g = \sup_B \int_{\mathbb{R}} \sigma \chi_I R_\alpha(\chi_I g) \\ &= \sup_B \int_0^\infty \sigma(\{x \in I : R_\alpha(\chi_I g) > \lambda\}) d\lambda \\ &\leq C \int_0^\infty \min\{\sigma(I), \lambda^{-p'}\} d\lambda \\ &\leq C \sigma(I)^{\frac{1}{p}}. \end{aligned}$$

But Theorem 1 for the operator N^- in the particular case $h \equiv 1$ and $k(y, x) = (x - y)^{\alpha-1}$ tells us that the weak type of R_α from $L^{q'}$ to $L^{p'}$ with respect to the weights $(u^{1-q'}, v^{1-p'})$ is equivalent to the condition

$$\left(\int_{-\infty}^a \frac{u(y)}{(b-y)^{(1-\alpha)q}} dy \right)^{\frac{1}{q}} \left(\int_a^b \sigma \right)^{\frac{1}{p'}} \leq C, \quad 1 < p < q.$$

Since it is an easy exercise to check that this condition is also necessary (even if $p = q$), the proof is finished.

Final remarks. (1) Condition (2.1) characterizes the weak type of the Weyl fractional integral if $1 \leq p < q < \infty$. We have proved it is actually equivalent to the apparently weaker condition: There exists C such that for any $a < b < c$ for which $\int_a^c u \leq 3 \int_a^b u$

$$\left(\int_a^b u \right)^{1/q} \left(\int_b^c \frac{v^{1-p'}(y)}{(y-a)^{(1-\alpha)p'}} dy \right)^{1/p'} \leq C.$$

(2) Even though we have proved that (1.3) $\implies$ (2.1), the proof of Theorem 2 is needed because Theorem 1 does not include the case $p = q$.

(3) The case $1 = p = q$ remains open even in the case of the two-sided fractional integral.

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