CONJUGATE POINTS IN $\mathcal{D}_\mu(T^2)$

GERARD MISIOLEK

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ABSTRACT. An example of a geodesic in $\mathcal{D}_\mu(T^2)$ with conjugate points is given, thus providing an affirmative answer to a question of V.I. Arnol'd.

1. INTRODUCTION

Let $\mathcal{D}_\mu(T^2)$ be the group of smooth volume preserving diffeomorphisms of a 2 torus equipped with a locally euclidean metric $\langle, \rangle$. It is convenient to enlarge it to include all such diffeomorphisms which are of Sobolev class H^s . If $s > 2$ this new set $\mathcal{D}_\mu^s(T^2)$, as demonstrated by Ebin and Marsden [EM], can be given a structure of an infinite dimensional submanifold of a weak Riemannian manifold $\mathcal{D}^s(T^2)$, the group of all H^s diffeomorphisms of T^2 . The tangent space to $\mathcal{D}^s(T^2)$ ($\mathcal{D}_\mu^s(T^2)$) at a point η consists of all (divergence free) H^s vector fields on T^2 which cover η . The weak Riemannian structure is given by the L^2 inner product

$$(1.1) \quad (X, Y)_{L^2} = \int_{T^2} \langle X(x), Y(x) \rangle_{\eta(x)} dx$$

where $X, Y \in T_\eta \mathcal{D}^s(T^2)$.

For an arbitrary H^s vector field X on T^2 if Δ denotes the Laplacian of the euclidean metric $\langle, \rangle$, then letting f be a solution of the equation

$$\Delta f = \operatorname{div} X$$

we obtain an L^2 orthogonal decomposition of X into the divergence free and gradient parts

$$X = (X - \operatorname{grad} f) + \operatorname{grad} f.$$

Since for each $\eta \in \mathcal{D}_\mu^s(T^2)$ the right translation $R_\eta(\xi) = \xi \circ \eta$ is an isometry of (1.1), this induces a splitting of $T_\eta \mathcal{D}^s(T^2)$ into a direct sum

$$(1.2) \quad T_\eta \mathcal{D}^s(T^2) = T_\eta \mathcal{D}_\mu^s(T^2) \oplus_{L^2} \operatorname{grad} H^{s+1}(T^2) \circ \eta.$$

We will denote by P_η and Q_η the projections onto the first and the second summands respectively.

The metric (1.1) induces on $\mathcal{D}^s(T^2)$ and $\mathcal{D}_\mu^s(T^2)$ right invariant Levi Civita connections $\bar{\nabla}$ and $\tilde{\nabla} = P_\eta \bar{\nabla}$ which have smooth exponential maps. The geodesics

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$t \rightarrow \xi(t)$ of $\tilde{\nabla}$ correspond to periodic motions of an ideal fluid in R^2 and can be obtained from a variational principle as stationary points of the L^2 action integral

$$(1.3) \quad E(\xi) = \frac{1}{2} \int_0^l |\dot{\xi}(t)|_{L^2}^2 dt$$

Using the splitting (1.2) the Euler-Lagrange equations of (1.3) are often written as a nonlinear partial differential equation

$$(1.4) \quad \begin{aligned} \partial_t V(t) + \nabla_{V(t)} V(t) &= \text{grad } p(t), \\ \text{div } V(t) &= 0 \end{aligned}$$

where $V(t) = \dot{\xi}(t) \circ \xi^{-1}(t)$ is a time dependent vector field on T^2 and $p(t)$ is the pressure function which can be determined from $V(t)$.

The idea that studying the geometry of $\mathcal{D}_\mu(T^2)$ is important for hydrodynamics goes back to Arnol'd [A], where he investigated stability of ideal fluids and presented complete computations of the curvature tensor of $\mathcal{D}_\mu(T^2)$ at the identity e . He showed that in many directions the sectional curvature was negative and gave an example of a two-plane for which it was positive. Arnol'd then asked whether there were any conjugate points in $\mathcal{D}_\mu(T^2)$. In this paper we show that a modification of Arnol'd's example yields a geodesic in $\mathcal{D}_\mu(T^2)$ which provides an affirmative answer to this question.

Namely, on T^2 consider the function $\phi = \frac{-1}{\pi\sqrt{40}} \cos 6x_1 \cos 2x_2$. Let $V = \text{rot } \phi$ be the corresponding divergence free vector field. We will prove the following

Theorem 1. *Let $\eta(t)$ be a geodesic in $\mathcal{D}_\mu^s(T^2)$ emanating from $\eta(0) = e$ in the direction of $\dot{\eta}(0) = V$. There exist $k > 0$ and $t_c \in (0, \pi\sqrt{\frac{2}{k}}]$ such that $\eta(t_c)$ is conjugate to e along η .*

For a general Hilbert Riemannian manifold $\mathcal{M}$ two types of conjugate points are possible. If $\xi(t)$ is a geodesic in $\mathcal{M}$ and $\exp^\mathcal{M}$ is the exponential map of the (possibly weak) Riemannian metric, we say that the point $\xi(l)$ is epiconjugate (monoconjugate) to $\xi(0)$ along ξ if the map $\exp_{\xi(0)*l\dot{\xi}(0)}^\mathcal{M} : T_{l\dot{\xi}(0)} T_{\xi(0)}\mathcal{M} \rightarrow T_{\xi(l)}\mathcal{M}$ is not onto (one to one).¹ We shall say that a point is conjugate if it is either epi- or monoconjugate. In finite dimensions the two types coincide.

Existence of conjugate points in $\mathcal{D}_\mu^s(M)$ is related to stability of fluid flows in M . The method presented in this paper may be used to find conjugate points along geodesics corresponding to flows on arbitrary compact Riemannian manifold M .

2. OUTLINE OF THE PROOF

We begin with some basic facts. Let $\xi(s, t) : (-\epsilon, \epsilon) \times [0, l] \rightarrow \mathcal{D}_\mu^s(T^2)$ be a two parameter variation of a geodesic $\xi(t)$ with $\xi(s, 0) = \xi(0) = e$ and $\xi(s, l) = \xi(l)$, and let $Z(t) = \partial_s \xi(s, t)|_{s=0}$ be the associated vector field. Then the first and the second variations of the L^2 action (1.3) are

$$(2.5) \quad \begin{aligned} E'(\xi)_0^l(Z) &= (Z, \dot{\xi})_0^l - \int_0^l (Z, \tilde{\nabla}_\xi \dot{\xi}) dt = 0, \\ E''(\xi)_0^l(Z, Z) &= \int_0^l \{(\tilde{\nabla}_\xi Z, \tilde{\nabla}_\xi Z) - (\tilde{R}_\xi(Z, \dot{\xi})\dot{\xi}, Z)\} dt \end{aligned}$$

¹This definition was given by Grossman [G].

where $\tilde{R}_\xi$ is the curvature tensor of the L^2 metric on $\mathcal{D}_\mu^s(T^2)$.

We will also need

Gauss Lemma. *If $X \in T_e \mathcal{D}_\mu^s(T^2)$ and $Y \in T_{tX} T_e \mathcal{D}_\mu^s(T^2)$ is such that $(Y, X) = 0$, then $(\text{exp}_{e*tX} X, \text{exp}_{e*tX} Y) = 0$.*

The proof of this lemma follows readily, as in [CE], pp. 8–9, from the first variation formula in (2.5).

Let $V = \text{rot } \phi$ be the vector field in Theorem 1. We first note that $P_e \nabla_V V = 0$. This implies that V is a stationary (that is, independent of time) solution of (1.4) and hence $\eta(t)$ is a one parameter subgroup of $\mathcal{D}_\mu^s(T^2)$.

Next, let $W = \text{rot } \psi$ be a divergence free vector field on T^2 corresponding to the function $\psi = -\cos(mx_1 + x_2) - \cos(mx_1 - 3x_2)$, m an integer. A lengthy but straightforward computation shows that the expression

$$(2.6) \quad \frac{1}{|W|_{L^2}^2} (\nabla_{[V,W]} V + \nabla_V [V, W], W)$$

can be estimated from below by a positive constant k whenever $m = \pm 4, \pm 5, \pm 6$ or ± 7 . Let $t_k = \pi \sqrt{\frac{2}{k}}$.

Lemma 2. *Let $\eta(t)$ be the geodesic in the theorem. There exists a vector field $\tilde{W}(t)$ along η vanishing at e and $\eta(t_k)$ for which the bilinear form $E''(\eta)_0^{t_k}(\tilde{W}, \tilde{W}) < 0$.*

Proof. Let $f(t)$ be a nonzero function such that $f(0) = f(t_k) = 0$. Right translate $W \in T_e \mathcal{D}_\mu(T^2)$ to a vector field along η and define $\tilde{W}(t) = f(t) \cdot W \circ \eta$. A convenient formula for the connection $\tilde{\nabla}$ can be obtained from (1.4). Applied to $\tilde{W}(t)$ in the direction $\dot{\eta}$, it gives

$$\begin{aligned} \tilde{\nabla}_{\dot{\eta}} \tilde{W} &= P_\eta \tilde{\nabla}_{\dot{\eta}} \tilde{W} = P_\eta \left(\frac{d}{dt} (\tilde{W} \circ \eta^{-1}) \circ \eta + (\nabla_{\dot{\eta} \circ \eta^{-1}} \tilde{W} \circ \eta^{-1}) \circ \eta \right) \\ &= f' \cdot W \circ \eta + f \cdot (P_e \nabla_V W) \circ \eta. \end{aligned}$$

Thus we obtain

$$\begin{aligned} (\tilde{\nabla}_{\dot{\eta}} \tilde{W}, \tilde{\nabla}_{\dot{\eta}} \tilde{W}) &= f'^2 |W|_{L^2}^2 + 2ff'(W, P_e \nabla_V W) + f^2 |P_e \nabla_V W|_{L^2}^2 \\ &= f'^2 |W|_{L^2}^2 + f^2 \|P_e \nabla_V W\|_{L^2}^2 \end{aligned}$$

since

$$(W, P_e \nabla_V W) = (W, \nabla_V W) = \frac{1}{2} (V, \text{grad} \langle W, W \rangle) = 0$$

because both V and W are divergence free.

On the other hand we have $(\tilde{R}_\eta(\tilde{W}, \dot{\eta})\dot{\eta}, \tilde{W}) = f^2(\tilde{R}_e(W, V)V, W)$ by the right invariance of the weak metric and connection $\tilde{\nabla}$ and $\dot{\eta} = V \circ \eta$. Since $\mathcal{D}_\mu^s(T^2)$ is a Riemannian submanifold of the full diffeomorphism group $\mathcal{D}^s(T^2)$, their curvature tensors $\tilde{R}_e$ and $\tilde{R}_e$ respectively are related by the Gauss-Codazzi equations. Moreover, for any smooth compact Riemannian manifold M , the curvature tensor of $\mathcal{D}^s(M)$ is completely determined by the curvature tensor R of the manifold M . Thus from (1.2) and the equations of Gauss-Codazzi together with the estimate on

(2.6) and the fact that $P_e \nabla_V V = 0$ we get

$$\begin{aligned}
& E''(\eta)_0^{t_k}(\tilde{W}, \tilde{W}) \\
&= \int_0^{t_k} \{f'^2 |W|_{L^2}^2 + f^2 |P_e \nabla_V W|_{L^2}^2 \\
&\quad - f^2 (R(W, V)V, W) - f^2 (\nabla_V V, \nabla_W W) + f^2 |Q_e \nabla_V W|_{L^2}^2\} dt \\
&= \int_0^{t_k} \{f'^2 |W|_{L^2}^2 - f^2 (\nabla_{[V, W]} V + \nabla_V [V, W], W)\} dt \\
&\leq |W|_{L^2}^2 \int_0^{t_k} \{f'^2 - k f^2\} dt.
\end{aligned}$$

If we now take $f(t) = \sin t \sqrt{\frac{k}{2}}$, then $\tilde{W}(0) = \tilde{W}(t_k) = 0$ and $E''(\eta)_0^{t_k}(\tilde{W}, \tilde{W}) \leq -\frac{\pi}{2} \sqrt{\frac{k}{2}} < 0$. $\square$

Since $(W, V) = 0$, the field $\tilde{W}(t)$ constructed above is in fact perpendicular to $\dot{\eta}(t)$ in the metric (1.1) by right invariance.

To proceed we need the following

Lemma 3. *Let $\eta(t)$ be as above. Let $Z(t)$ be a smooth vector field on η such that $Z(0) = Z(t_k) = 0$. If there are no points conjugate to e along $\eta(t)$ for $0 \leq t \leq t_k$, then $E''(\eta)_0^{t_k}(Z, Z) \geq 0$.*

Proof. Let $\tilde{R}_t = \tilde{R}_\eta(\cdot, \dot{\eta})\dot{\eta} : T_\eta \mathcal{D}_\mu^s(T^2) \rightarrow T_\eta \mathcal{D}_\mu^s(T^2)$ and τ_t denote the curvature operator and the parallel translation of the weak metric (1.1) along η . Let $U(t)$ be the evolution operator of the Jacobi equation along η written as a first order system in $T_e \mathcal{D}_\mu^s(T^2) \times T_e \mathcal{D}_\mu^s(T^2)$

$$\frac{d}{dt} \begin{pmatrix} Y \\ Y' \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ (\tau_t)^{-1} \circ \tilde{R}_t \circ \tau_t & 0 \end{pmatrix} \begin{pmatrix} Y \\ Y' \end{pmatrix} = 0.$$

Since for any t the map $X \rightarrow (\tau_t)^{-1} \circ \tilde{R}_t \circ \tau_t X$ is linear and bounded in the (strong) H^s topology of $T_e \mathcal{D}_\mu^s(T^2)$, $s > 2$ (cf. [M]), the operator $U(t)$ is a topological isomorphism of $T_e \mathcal{D}_\mu^s(T^2) \times T_e \mathcal{D}_\mu^s(T^2)$. Thus the linear map²

$$T_{tV} T_e \mathcal{D}_\mu^s(T^2) \ni X \rightarrow \exp_{e*} tV X = \tau_t \circ \pi_1 \circ U(t)(0, X) \in T_{\eta(t)} \mathcal{D}_\mu^s(T^2)$$

is bounded. The assumption about conjugate points implies it is one to one and onto and therefore also an isomorphism for each t by the open mapping theorem. It follows that $\exp_e$ is a diffeomorphism on an open neighbourhood $\mathcal{U}(t)$ of each tV in $T_e \mathcal{D}_\mu^s(T^2)$. By compactness of $[0, t_k]$ we can find a finite number of such neighbourhoods $\mathcal{U}(t_1), \dots, \mathcal{U}(t_N)$ whose union $\mathcal{U} = \bigcup_{i=1}^N \mathcal{U}(t_i)$ covers the set $\{tV : 0 \leq t \leq t_k\}$ and such that $\eta([0, t_k]) \subset \bigcup_{i=1}^N \exp_e \mathcal{U}(t_i) = \mathcal{V}$.

Similarly by compactness it is possible to choose a $\delta > 0$ so that each $\exp_{\eta(t)}$ maps a ball $B^s(\delta)$ of radius δ in $T_{\eta(t)} \mathcal{D}_\mu^s(T^2)$ diffeomorphically onto $\exp_{\eta(t)} B^s(\delta) \subset \mathcal{V}$ for $t \in [0, t_k]$.

Let $\xi(s, t) = \exp_{\eta(t)} sZ(t)$. Then for a fixed s , satisfying $s < \frac{\delta}{\max_t |Z(t)|_{H^s}}$, the curve $t \rightarrow \xi(s, t)$ lies in $\mathcal{V}$. Consequently $c_s(t) = \exp_e^{-1} \xi(s, t)$ is a curve in $\mathcal{U}$ joining

²Here π_1 denotes the projection onto the first summand of $T_e \mathcal{D}_\mu^s \times T_e \mathcal{D}_\mu^s$.

$0 \in T_e \mathcal{D}_\mu^s(T^2)$ and $t_k V$. If we set $c_s(t) = r(t) \frac{c(t)}{|c(t)|_{L^2}}$ where $r : [0, t_k] \rightarrow R$ is a function satisfying $r(0) = 0$ and $r(t_k) = t_k$, we obtain

$$\dot{c}_s(t) = \dot{r}(t) \frac{c_s(t)}{|c_s(t)|_{L^2}} + r(t) \frac{d}{dt} \left(\frac{c_s(t)}{|c_s(t)|_{L^2}} \right).$$

It follows from the Gauss Lemma that

$$|\dot{\xi}_s(t)|_{L^2}^2 = \left| \frac{d}{dt} (\text{e}\tilde{\text{x}}\text{p}_e c_s(t)) \right|_{L^2}^2 = |\text{e}\tilde{\text{x}}\text{p}_{e*c_s(t)} \dot{c}_s(t)|_{L^2}^2 \geq (\dot{r}(t))^2.$$

Therefore by the Cauchy-Schwartz inequality $E(\xi_s) \geq \frac{1}{2} \int_0^{t_k} (\dot{r}(t))^2 dt \geq \frac{1}{2} t_k = E(\eta)$ for $s < \frac{\delta}{\max |Z(t)|_{H^s}}$, but this implies that $E''(\eta)(Z, Z) \geq 0$. $\square$

The theorem is now a consequence of the above lemmas.

3. CONCLUDING REMARKS

Remark 1. A simple corollary of the above theorem is that conjugate points can be found on $\mathcal{D}_\mu^s(T^n)$ where T^n is a flat torus of arbitrary dimension n .

Remark 2. More examples of conjugate points in $\mathcal{D}_\mu^s(M)$ can be found if M has positive curvature. For instance, any rotation of a sphere in R^3 yields a stationary solution of the Euler equations (1.4). Sectional curvatures along the corresponding geodesic $\eta(t)$ in $\mathcal{D}_\mu^s(S^2)$ are nonnegative and $\eta(t)$ can be shown to have monoconjugate points.

Remark 3. In the light of Arnol'd's curvature computations mentioned in the Introduction it is natural to expect that there exist in $\mathcal{D}_\mu^s(T^2)$ geodesics without conjugate points. Indeed, this is shown to be true (cf. [M]) of any geodesic in $\mathcal{D}_\mu^s(M)$ which is also a geodesic in $\mathcal{D}^s(M)$ (such geodesics correspond to fluid flows in M with constant pressure) whenever M is a Riemannian manifold of nonpositive sectional curvature.

An example on a flat 2-torus is provided by

$$\eta(t)(x_1, x_2) = (x_1 + th(x_2), x_2)$$

where h is an arbitrary smooth periodic function.

Remark 4. One may apply the method presented here to study stability of specific fluid flows. A particularly interesting example on a flat 3-torus studied by Arnol'd [A] and more recently by Friedlander, Gilbert and Vishik [FGV] and Dombre et al. [DFGHMS] is given by a vector field

$$V = (A \sin x_3 + C \cos x_2, B \sin x_1 + A \cos x_3, C \sin x_2 + B \cos x_1)$$

where A , B and C are arbitrary constants. This flow is stationary and there exist two-planes at the identity of $\mathcal{D}_\mu^s(T^3)$ containing V for which the curvature is positive. It is likely that conjugate points can be found along this flow as well.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, INDIANA 46556
E-mail address: misiolek.1@nd.edu