A PROBLEM ON PRODUCTS OF TOEPLITZ OPERATORS

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ABSTRACT. A natural and interesting problem on classical Hardy space of one complex variable is the following:

Problem: If $T_{\varphi_1}T_{\varphi_2}\cdots T_{\varphi_n} = 0$, then there exist some i such that $\varphi_i = 0$.

In this note, we establish the kernel inclusion theorem for the products of Toeplitz operators. Using this fact, in case $n = 5$, we give the above question an affirmative answer.

Let D be the open unit disk in the complex plane and T its boundary. For the definitions and related conceptions of Hardy space $H^2(T, d\theta/2\pi)$ and Toeplitz operators on $H^2(T, d\theta/2\pi)$, refer to [1, Chapters 6 and 7]; for convenience, write $H^2(T, d\theta/2\pi)$ as H^2 . To prove the main results in this note, we establish the following lemmas.

Lemma 1. For $g \in L^\infty$ and $g \neq 0$, let E be a measurable subset of T with $0 < \text{meas } E < 2\pi, g|_E = 0$. Then $\ker T_g = \{0\}$.

The proof of Lemma 1 is trivial.

Lemma 2. For $f, g \in L^\infty$ and $f, g \neq 0$, let E be a measurable subset of T with $0 < \text{meas } E < 2\pi$ and $g|_E = 0$. If $\ker T_f T_g \neq \{0\}$, then there exists a function $\psi (\neq 0)$ such that

$$T_g h = \psi h \quad \forall h \in \ker T_f T_g.$$

Proof. For $h_0 \in \ker T_f T_g, h_0 \neq 0$, by Lemma 1, there exist $h_1 \in H^2$ and $h_1 \neq 0$ such that $f T_g h_0 = \overline{h_1}$. So, $T_g h_0 = \frac{\overline{h_1}}{f}$. It follows that there is $h_2 \in H^2$ such that $g h_0 = \frac{\overline{h_1}}{f} - \overline{h_2}$. Hence $\frac{1}{f}|_E = (\frac{\overline{h_2}}{h_1})|_E$. Similarly for $h'_0 \in \ker T_f T_g$ and $h'_0 \neq h_0$, there exist $h'_1, h'_2 \in H^2$ such that $\frac{1}{f}|_E = (\frac{\overline{h'_2}}{h'_1})|_E$. Therefore

$$\left(\frac{\overline{h_2}}{h_1}\right)\Big|_E = \left(\frac{\overline{h'_2}}{h'_1}\right)\Big|_E$$

i.e., $(h_2 h'_1 - h'_2 h_1)|_E = 0$. Since $h_2 h'_1 - h'_2 h_1 \in H^1$ and $0 < \text{meas } E < 2\pi$, we get that $h_2 h'_1 = h'_2 h_1$. Put $h' = \frac{h_2}{h_1} = \frac{h'_2}{h'_1}$; according to the preceding reasoning, we

obtain

$$\begin{aligned} gh_0 &= \frac{\overline{h_1}}{f} - \overline{h_2} = \frac{\overline{h_1}}{f} - \overline{h_1 h'} = \overline{h_1} \left(\frac{1}{f} - \overline{h'} \right) \\ &= f \left(\frac{1}{f} - \overline{h'} \right) T_g h_0 = (1 - f \overline{h'}) T_g h_0. \end{aligned}$$

Similarly, we have

$$gh'_0 = (1 - f \overline{h'}) T_g h'_0.$$

Since $\frac{gh_0}{gh'_0}|_{T \setminus E} = \frac{h_0}{h'_0}|_{T \setminus E}$, the above argument leads to the following:

$$\frac{h_0}{h'_0} = \frac{T_g h_0}{T_g h'_0},$$

i.e., $\frac{T_g h_0}{h_0} = \frac{T_g h'_0}{h'_0}$. Set $\psi = \frac{T_g h_0}{h_0}$; clearly such a ψ is independent on the choice of h_0 in the $\ker T_f T_g$. Thus the following is true:

$$T_g h = \psi h \quad \forall h \in \ker T_f T_g.$$

This completes the proof of Lemma 1. $\square$

Next, we establish the kernel inclusion theorem, which is interesting in itself.

Theorem 1. *Under the same assumption as in Lemma 2, there exist $\varphi \in L^\infty$ and $\varphi \neq 0$ such that*

$$\ker T_f T_g \subset \ker T_\varphi.$$

Proof. First claim $\ker T_f T_g \otimes \overline{H^2} \neq L^1$, where $\ker T_f T_g \otimes \overline{H^2}$ denotes the L^1 -closure of linear manifold $\{\text{finite sum } \sum \lambda_i h_i \overline{h'_i} : \forall h_i \in \ker T_f T_g, \forall h'_i \in H^2\}$. If not, by Lemma 2 there exists a function $\psi (\neq 0)$ such that for any $h \in \ker T_f T_g$ we have $T_g h = \psi h$. Since

$$\begin{aligned} \int g \left(\sum \lambda_i h_i \overline{h'_i} \right) d\theta &= \sum \lambda_i \langle T_g h_i, h'_i \rangle = \sum \lambda_i \langle \psi h_i, h'_i \rangle \\ &= \int \psi \left(\sum \lambda_i h_i \overline{h'_i} \right) d\theta, \end{aligned}$$

it follows that

$$\left| \int \psi \left(\sum \lambda_i h_i \overline{h'_i} \right) d\theta \right| \leq \|g\|_\infty \left\| \sum \lambda_i h_i \overline{h'_i} \right\|_{L^1}.$$

Because the above inequality is true on the dense subset $\ker T_f T_g \otimes \overline{H^2}$ of L^1 , M_ψ , determines a bounded linear functional on L^1 . From this, we obtain a $\psi \in L^\infty$ and $\psi = g$. By Lemma 1, $|\psi| > 0$ a.e. This contradicts $g|_E = 0$. It follows that $\ker T_f T_g \otimes \overline{H^2} \neq L^1$. We prove the claim.

According to the claim there exists a bounded linear functional which annihilates $\ker T_f T_g \otimes \overline{H^2}$, i.e., there exist $\varphi \in L^\infty$ and $\varphi \neq 0$ such that for any $h_0 \in \ker T_f T_g$, $h \in H^2$, we have

$$\int \varphi h_0 \overline{h} d\theta = \langle T_\varphi h_0, h \rangle = 0.$$

Hence $h_0 \in \ker T_\varphi$, and it follows that $\ker T_f T_g \subset \ker T_\varphi$. This completes the proof of Theorem 1. $\square$

To prove Theorem 2, we need the following lemma.

Lemma 3 ([1, p. 179]). *If $T_{\varphi_1}T_{\varphi_2}\cdots T_{\varphi_n} = 0$, then $\varphi_1\varphi_2\cdots\varphi_n = 0$.*

Theorem 2. *If $T_{\varphi_1}T_{\varphi_2}T_{\varphi_3}T_{\varphi_4}T_{\varphi_5} = 0$, then there exist some i such that $\varphi_i = 0$.*

Proof. Suppose $\ker T_{\varphi_1} \neq 0$ and $\ker T_{\overline{\varphi_5}} \neq 0$ without loss of generality. Then clearly $|\varphi_1\varphi_5| > 0$ a.e. By Lemma 3 we have

$$\varphi_2\varphi_3\varphi_4 = 0.$$

Assume $\varphi_3 \neq 0$. It follows that there exists a measurable subset E of T with $0 < \text{meas } E < 2\pi$ such that

$$\varphi_2|_E = 0 \quad \text{or} \quad \varphi_4|_E = 0.$$

In case $\varphi_2|_E = 0$, suppose $T_{\varphi_3}T_{\varphi_4}T_{\varphi_5} \neq 0$; hence $\ker T_{\varphi_1}T_{\varphi_2} \neq \{0\}$. From Theorem 1 and $\text{Rang } T_{\varphi_3}T_{\varphi_4}T_{\varphi_5} \subset \ker T_{\varphi_1}T_{\varphi_2}$, we know that there exists a $\varphi \in L^\infty$, $\varphi \neq 0$, such that

$$(*) \quad T_\varphi T_{\varphi_3}T_{\varphi_4}T_{\varphi_5} = 0.$$

Since $|\varphi| > 0$, $|\varphi_5| > 0$ a.e., by Lemma 3 $\varphi_3\varphi_4 = 0$. Thus there exists a measurable subset E' of T with $0 < \text{meas } E' < 2\pi$ such that $\varphi_4|_{E'} = 0$. By (*), we have

$$T_{\overline{\varphi_5}}T_{\overline{\varphi_4}}T_{\overline{\varphi_3}}T_{\overline{\varphi}} = 0.$$

Since $T_{\overline{\varphi_3}}T_{\overline{\varphi}} \neq 0$, by Theorem 1 and $\text{Rang } T_{\overline{\varphi_3}}T_{\overline{\varphi}} \subset \ker T_{\overline{\varphi_5}}T_{\overline{\varphi_4}}$, there exists a $\varphi' \in L^\infty$, $|\varphi'| > 0$ a.e., such that

$$T_{\varphi'}T_{\overline{\varphi_3}}T_{\overline{\varphi}} = 0.$$

Again by Lemma 3, $\varphi_3 = 0$. This contradicts the assumption; thus we obtain

$$\varphi_2 = 0 \quad \text{or} \quad \varphi_4 = 0.$$

Let $\varphi_4|_E = 0$, and assume $\varphi_3 \neq 0$. Similarly, we have

$$\varphi_2 = 0 \quad \text{or} \quad \varphi_4 = 0.$$

Therefore if $T_{\varphi_1}T_{\varphi_2}T_{\varphi_3}T_{\varphi_4}T_{\varphi_5} = 0$, then there exists some i such that $\varphi_i = 0$. This completes the proof of Theorem 2. $\square$

Using the approach in this note, we may prove the following

Theorem 3. *If $T_{\varphi_1}T_{\varphi_2}T_{\varphi_3}T_{\varphi_4}T_{\varphi_5}$ has finite rank, then there exists some i such that $\varphi_i = 0$.*

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