SHARP MAXIMAL INEQUALITIES FOR STOCHASTIC INTEGRALS IN WHICH THE INTEGRATOR IS A SUBMARTINGALE

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ABSTRACT. We obtain sharp maximal inequalities for strong subordinates of real-valued submartingales. Analogous inequalities also hold for stochastic integrals in which the integrator is a submartingale. The impossibility of general moment inequalities is also demonstrated.

1. INTRODUCTION

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a right-continuous filtration $(\mathcal{F}_t)_{t \geq 0}$ where $\mathcal{F}_0$ contains all P -null sets. Suppose X is an adapted right-continuous real-valued submartingale with left limits and H is a predictable process with values in the closed unit ball of $\mathbb{R}^\nu$, where ν is a positive integer. Define an adapted right-continuous process Y with left limits by

$$Y_t = H_0 X_0 + \int_{(0,t]} H_s dX_s.$$

We will compare the size of Y with that of X by finding constants β such that for all $\lambda > 0$,

$$(1.1) \quad \lambda P(Y^* \geq \lambda) \leq \beta \|X\|_1$$

where $\|X\|_1 = \sup_{t \geq 0} \|X_t\|_1$ and $Y^* = \sup_{t \geq 0} |Y_t|$. In this paper we will denote the Euclidean norm of $y \in \mathbb{R}^\nu$ by $|y|$ and the inner product of $y, k \in \mathbb{R}^\nu$ by $y \cdot k$.

If we restrict X to the class of martingales, it is known that the best constant satisfying (1.1) is $\beta = 2$ [2, 3]. By the best constant we mean that for any $\beta < 2$ there exist a martingale X , a predictable process H , and a $\lambda > 0$ such that $\lambda P(Y^* \geq \lambda) > \beta \|X\|_1$. It is also known [5] that if we restrict X to the class of nonnegative submartingales, then the best constant satisfying (1.1) is $\beta = 3$.

In this paper we will show that for the class of real-valued submartingales, the best constant in (1.1) is $\beta = 6$. To do this we shall first prove the analogous inequality and more for discrete-time submartingales. In the last section of this

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paper we shall show that there are no moment inequalities of the form $\|Y\|_p \leq \beta \|X\|_p$ where $1 < p < \infty$ and β is finite and depends only on p . In fact, we shall show that for any $p \in [1, \infty)$, there is no finite β such that $\|Y\|_1 \leq \beta \|X\|_p$. For the case $p = \infty$, see [7] where it is shown that if $\|X\|_\infty = 1$, then there is a constant γ such that for $\lambda > 4$, $P(Y^* \geq \lambda) \leq \gamma \exp(-\lambda/4)$, so, for any $r \in [1, \infty)$, $\|Y\|_r$ is bounded by some constant depending only on r .

2. A MAXIMAL INEQUALITY FOR SUBMARTINGALES

Let $f_0, f_1, \dots$ be a real-valued submartingale relative to a filtration $(\mathcal{F}_n)_{n \geq 0}$ on a probability space $(\Omega, \mathcal{F}, P)$ with difference sequence $d_0, d_1, \dots$, and $g_0, g_1, \dots$ an $\mathbb{R}^\nu$ -valued process adapted to $(\mathcal{F}_n)_{n \geq 0}$ with difference sequence $e_0, e_1, \dots$, where ν is a positive integer. We say that g is strongly subordinate to f if g is both differentially subordinate and conditionally differentially subordinate to f , i.e. for all $n \geq 0$, $|e_n| \leq |d_n|$ and $|\mathbf{E}(e_{n+1}|\mathcal{F}_n)| \leq |\mathbf{E}(d_{n+1}|\mathcal{F}_n)|$. Note that if for $k \geq 0$, $e_k = h_k d_k$ where $h_k : \Omega \rightarrow [-1, 1]$ is $\mathcal{F}_{k-1}$ -measurable, then g is strongly subordinate to f . In particular, if g is a ± 1 -transform of f , i.e. $e_k = \epsilon_k d_k$ where $\epsilon_k \in \{-1, 1\}$, then g is strongly subordinate to f .

Theorem 2.1. *If $f = (f_n)_{n \geq 0}$ is a submartingale relative to a filtration $(\mathcal{F}_n)_{n \geq 0}$ and $g = (g_n)_{n \geq 0}$ is strongly subordinate to f , then for all $\lambda > 0$,*

$$(2.1) \quad \lambda P(g^* \geq \lambda) \leq \lambda P(\sup_{n \geq 0} (|f_n| + |g_n|) \geq \lambda) \leq 4 \sup_{n \geq 0} \mathbf{E}f_n^+ - 2\mathbf{E}f_0$$

where $g^* = \sup_{n \geq 0} |g_n|$.

Remarks. If f is a martingale, then $\mathbf{E}f_n^+$ and $\mathbf{E}f_n^-$ are nondecreasing sequences. It then follows from $\mathbf{E}f_0 = \mathbf{E}f_n^+ - \mathbf{E}f_n^-$ that $\|f\|_1 = 2 \sup_{n \geq 0} \mathbf{E}f_n^+ - \mathbf{E}f_0$, where $\|f\|_1 = \sup_{n \geq 0} \|f_n\|_1$. Thus in the martingale case, (2.1) implies that

$$\lambda P(g^* \geq \lambda) \leq \lambda P(\sup_{n \geq 0} (|f_n| + |g_n|) \geq \lambda) \leq 2\|f\|_1$$

which is Theorem 4.1 of [4]. If f is a nonnegative supermartingale, (2.1) implies

$$\lambda P(g^* \geq \lambda) \leq \lambda P(\sup_{n \geq 0} (|f_n| + |g_n|) \geq \lambda) \leq 2\mathbf{E}f_0$$

which is Theorem 8.1 of [5]. Both results are shown to be sharp in the articles quoted. If f is a nonnegative submartingale with $f_0 = 0$, the resulting inequality is not sharp in the case $f_0 = 0$, as can be seen from Theorem 4.1 of [5] which shows in this case

$$\lambda P(g^* \geq \lambda) \leq \lambda P(\sup_{n \geq 0} (|f_n| + |g_n|) \geq \lambda) \leq 3\|f\|_1.$$

Proof. We will assume $\|f\|_1$ is finite. This is equivalent to saying $\sup_{n \geq 0} \mathbf{E}f_n^+$ is finite, as for all $n \geq 0$, $\mathbf{E}f_n^+ \leq \|f_n\|_1 \leq 2\mathbf{E}f_n^+ - \mathbf{E}f_0$. The first inequality is obvious, the second follows from $\mathbf{E}f_0 \leq \mathbf{E}f_n = \mathbf{E}f_n^+ - \mathbf{E}f_n^-$.

To show (2.1), it suffices to show that for $n \geq 0$,

$$(2.2) \quad \lambda P(|f_n| + |g_n| \geq \lambda) \leq 4\mathbf{E}f_n^+ - 2\mathbf{E}f_0$$

since if (2.2) holds, then with $\tau = \inf\{n \geq 0 : |f_n| + |g_n| \geq \lambda\}$, τ is a stopping time, f^τ is a submartingale, and g^τ is strongly subordinate to f^τ , so by (2.2)

$$\lambda P(\sup_{m \leq n} (|f_m| + |g_m|) \geq \lambda) = \lambda P(|f_{\tau \wedge n}| + |g_{\tau \wedge n}| \geq \lambda) \leq 4\mathbf{E}f_{\tau \wedge n}^+ - 2\mathbf{E}f_0.$$

Since $(f_n^+)_{n \geq 0}$ is a submartingale, it follows by Doob's optional sampling theorem that $\mathbf{E}f_{\tau \wedge n}^+ \leq \mathbf{E}f_n^+$, thus implying (2.1).

By dividing by λ throughout in (2.2), we may assume $\lambda = 1$. Using the methods developed by Burkholder [2], we define V on $\mathbb{R} \times \mathbb{R}^\nu$ by

$$V(x, y) = \begin{cases} 1 - 4x^+, & \text{if } |x| + |y| \geq 1, \\ -4x^+, & \text{if } |x| + |y| < 1. \end{cases}$$

Then (2.2) is equivalent to $\mathbf{E}V(f_n, g_n) \leq -2\mathbf{E}f_0$. Define U on $\mathbb{R} \times \mathbb{R}^\nu$ by

$$U(x, y) = \begin{cases} 1 - 4x^+, & \text{if } |x| + |y| \geq 1, \\ |y|^2 - x^2 - 2x, & \text{if } |x| + |y| < 1. \end{cases}$$

Then $V \leq U$ (in the case of $|x| + |y| < 1$ this follows from $-4x^+ \leq -x^2 - 2x$ for $|x| < 1$) and $U(f_0, g_0) \leq -2f_0$ (recall that by assumption $|f_0| \geq |g_0|$).

Thus $\mathbf{E}V(f_n, g_n) \leq \mathbf{E}U(f_n, g_n)$ and $\mathbf{E}U(f_0, g_0) \leq -2\mathbf{E}f_0$. To show (2.2), it will suffice to show that for $1 \leq j \leq n$,

$$(2.3) \quad \mathbf{E}U(f_j, g_j) \leq \mathbf{E}U(f_{j-1}, g_{j-1}).$$

Define ϕ, ψ on $\mathbb{R} \times \mathbb{R}^\nu$ by

$$\phi(x, y) = \begin{cases} -4, & \text{if } |x| + |y| \geq 1 \text{ and } x \geq 0, \\ 0, & \text{if } |x| + |y| \geq 1 \text{ and } x < 0, \\ -2x - 2, & \text{if } |x| + |y| < 1, \end{cases}$$

$$\psi(x, y) = \begin{cases} 0, & \text{if } |x| + |y| \geq 1, \\ 2y, & \text{if } |x| + |y| < 1. \end{cases}$$

Then $U_x(x, y) = \phi(x, y)$ and $U_y(x, y) = \psi(x, y)$ for $|x| + |y| \neq 1$, $y \neq 0$, and $x \neq 0$ where $U_x(x, y)$ and $U_y(x, y)$ are the partials of U with respect to x and y respectively. Note that $|\psi| \leq -\phi$.

Claim: Given $h \in \mathbb{R}$ and $k \in \mathbb{R}^\nu$ with $|k| \leq |h|$, then for all $x \in \mathbb{R}$ and $y \in \mathbb{R}^\nu$

$$(2.4) \quad U(x + h, y + k) \leq U(x, y) + \phi(x, y)h + \psi(x, y) \cdot k.$$

This can be verified by checking the various cases:

For $|x| + |y| \geq 1$ and $x \geq 0$, we need to show $U(x + h, y + k) \leq 1 - 4(x + h)$. For $|x + h| + |y + k| \geq 1$ this is clear. For $|x + h| + |y + k| < 1$ it follows from

$$|y + k|^2 < (1 - |x + h|)^2 \leq 1 - 2(x + h) + (x + h)^2.$$

For $|x| + |y| \geq 1$ and $x < 0$, we need to show $U(x + h, y + k) \leq 1$. However $U(x, y) \leq 1$ for all x, y , this being obvious for $|x| + |y| \geq 1$. In the region $|x| + |y| < 1$, since $U_x(x, y) \leq 0$, it follows that $U(x, y) \leq |y|^2 - (|y| - 1)^2 - 2(|y| - 1) = 1$.

For the case $|x| + |y| < 1$, (2.4) is equivalent to showing

$$(2.5) \quad U(x+h, y+k) \leq |y+k|^2 - (x+h)^2 - 2(x+h) - |k|^2 + h^2.$$

For $|x+h| + |y+k| < 1$, this follows from $|k| \leq |h|$ and the definition of U . For $|x+h| + |y+k| \geq 1$, (2.5) can be rewritten as

$$(1 - |x+h|)^2 \leq |y+k|^2 + h^2 - |k|^2.$$

If $|x+h| \leq 1$, this inequality follows from $|k| \leq |h|$ and $|x+h| + |y+k| \geq 1$. If $|x+h| > 1$, then $(1 - |x+h|)^2 \leq (1 - |x| - |h|)^2$ and it suffices to show

$$(1 - |x|)^2 - 2|h|(1 - |x|) \leq |y|^2 - 2|y||k|.$$

Since $|h| \geq |k|$, it then suffices to show $(1 - |x|)^2 \leq |y|^2 + 2|h|(1 - |x| - |y|)$, an inequality which follows from $|x| + |h| \geq 1$ and $0 \leq |y|^2 - 2|y|(1 - |x|) + (1 - |x|)^2$, so that

$$(1 - |x|)^2 \leq |y|^2 + 2(1 - |x|)(1 - |x| - |y|) \leq |y|^2 + 2|h|(1 - |x| - |y|).$$

To prove (2.3), since $|e_j| \leq |d_j|$, by (2.4) we have

$$(2.6) \quad U(f_j, g_j) \leq U(f_{j-1}, g_{j-1}) + \phi(f_{j-1}, g_{j-1})d_j + \psi(f_{j-1}, g_{j-1}) \cdot e_j.$$

Since f is a submartingale, $\mathbf{E}(d_j | \mathcal{F}_{j-1}) \geq 0$. It then follows from $|\psi| \leq -\phi$ and g being strongly subordinate to f that

$$\phi(f_{j-1}, g_{j-1})\mathbf{E}(d_j | \mathcal{F}_{j-1}) + \psi(f_{j-1}, g_{j-1}) \cdot \mathbf{E}(e_j | \mathcal{F}_{j-1}) \leq 0.$$

Using this after taking the conditional expectations relative to $\mathcal{F}_{j-1}$ in (2.6) gives

$$\mathbf{E}(U(f_j, g_j) | \mathcal{F}_{j-1}) \leq U(f_{j-1}, g_{j-1}).$$

Taking expectations of both sides gives (2.3) and completes the proof.

3. DISCRETE-TIME SHARP MAXIMAL INEQUALITIES

Theorem 3.1. *If f is a submartingale relative to a filtration $(\mathcal{F}_n)_{n \geq 0}$ and g is strongly subordinate to f , then for all $\lambda > 0$*

$$(3.1) \quad \lambda P(g^* \geq \lambda) \leq \lambda P(\sup_{n \geq 0} (|f_n| + |g_n|) \geq \lambda) \leq 4 \|f\|_1 - 2\mathbf{E}f_0.$$

Thus if $f_0 \equiv 0$, then

$$(3.2) \quad \lambda P(g^* \geq \lambda) \leq \lambda P(\sup_{n \geq 0} (|f_n| + |g_n|) \geq \lambda) \leq 4 \|f\|_1,$$

while in general

$$(3.3) \quad \lambda P(g^* \geq \lambda) \leq \lambda P(\sup_{n \geq 0} (|f_n| + |g_n|) \geq \lambda) \leq 6 \|f\|_1.$$

The constants 4 and 6 are the best possible in (3.2) and (3.3) respectively, even in the case $\nu = 1$ and g is a ± 1 -transform of f .

Proof. The inequalities follow immediately from Theorem 2.1. For the sharpness, first consider the following example:

Example 3.1. Fix $3 < \beta < 4$ and let $\alpha = (4 - \beta)/4$, so that $\beta < 4 - 2\alpha$. On the Lebesgue interval $[0, 1]$, let $f_0 = g_0 \equiv 0$,

$$\begin{aligned} f_1 &= \mathbf{1}_{[0, \alpha)} - \frac{\alpha}{1 - \alpha} \mathbf{1}_{[\alpha, 1]}, & g_1 &= f_1, \\ f_2 &= f_1 - \mathbf{1}_{[\alpha, 2\alpha - \alpha^2)} + \frac{\alpha}{1 - \alpha} \mathbf{1}_{[2\alpha - \alpha^2, 1]}, & g_2 &= g_1 + \mathbf{1}_{[\alpha, 2\alpha - \alpha^2)} - \frac{\alpha}{1 - \alpha} \mathbf{1}_{[2\alpha - \alpha^2, 1]}, \\ f_3 &= f_2 + \frac{1}{1 - \alpha} \mathbf{1}_{[\alpha, 2\alpha - \alpha^2)}, & g_3 &= g_2 + \frac{1}{1 - \alpha} \mathbf{1}_{[\alpha, 2\alpha - \alpha^2)}. \end{aligned}$$

Then $f = (f_0, f_1, f_2, f_3)$ is a submartingale and $g = (g_0, g_1, g_2, g_3)$ is a ± 1 -transform of f . Note that $g_3 = \mathbf{1}_{[0, \alpha)} + 2\mathbf{1}_{[\alpha, 2\alpha - \alpha^2)} - (2\alpha/(1 - \alpha))\mathbf{1}_{[2\alpha - \alpha^2, 1]}$ and $f_3 = \mathbf{1}_{[0, \alpha)}$. Thus

$$2P(f_3 + g_3 \geq 2) = (4 - 2\alpha)\alpha > \beta \sup_{0 \leq j \leq 3} \mathbf{E}f_j^+.$$

Now let $\tilde{f}_0 = \tilde{g}_0 \equiv 0$ and for $j \geq 0$, $1 \leq k \leq 3$, and $s \in [0, 1]$, let

$$\begin{aligned} \tilde{f}_{3j+k}(s) &= \tilde{f}_{3j}(s) + \mathbf{1}_{[1-2^{-j}, 1-2^{-j-1}]}(s) f_k(2^{j+1}(s - 1 + 2^{-j})), \\ \tilde{g}_{3j+k}(s) &= \tilde{g}_{3j}(s) + \mathbf{1}_{[1-2^{-j}, 1-2^{-j-1}]}(s) g_k(2^{j+1}(s - 1 + 2^{-j})). \end{aligned}$$

By induction on $j \geq 0$, we have

$$\begin{aligned} P(\tilde{f}_{3j} = 1, \tilde{g}_{3j} = 1) &= (1 - 2^{-j})\alpha, & P(\tilde{f}_{3j} = 0, \tilde{g}_{3j} = 2) &= (1 - 2^{-j})(\alpha - \alpha^2), \\ P(\tilde{f}_{3j} = 0, \tilde{g}_{3j} = \frac{-2\alpha}{1 - \alpha}) &= (1 - 2^{-j})(1 - \alpha)^2, \end{aligned}$$

and, for $k \leq 3j$, $\text{supp} \tilde{f}_k \subseteq [0, 1 - 2^{-j}]$.

It follows that $\tilde{f}$ is a submartingale, $\tilde{g}$ is a ± 1 -transform of $\tilde{f}$, and, for $j \geq 0$, $1 \leq k \leq 3$, $\|\tilde{f}_{3j+k}\|_1 = \|\tilde{f}_{3j}\|_1 + 2^{-j-1}\|f_k\|_1$. Since $\|\tilde{f}_{3j}\|_1 = (1 - 2^{-j})\alpha$ and $\|f_1\|_1 = \|f_2\|_1 = 2\alpha$, we have that $\|\tilde{f}_{3j+k}\|_1 \leq \alpha = \mathbf{E}f_3$. Thus, with $\lambda = 2$,

$$\lim_{j \rightarrow \infty} \lambda P(\tilde{f}_{3j} + \tilde{g}_{3j} \geq \lambda) = \lambda P(f_3 + g_3 \geq \lambda) > \beta \mathbf{E}f_3 \geq \beta \sup_{k \geq 0} \|\tilde{f}_k\|_1.$$

Since we are assuming a strict inequality, there exists an n such

$$(3.4) \quad \lambda P(\tilde{f}_n + \tilde{g}_n \geq \lambda) > \beta \sup_{j \geq 0} \|\tilde{f}_j\|_1 \geq \beta \sup_{0 \leq j \leq n} \|\tilde{f}_j\|_1.$$

Now let $(r_j)_{j \geq 1}$ be a sequence of independent identically distributed random variables such that $P(r_1 = 1) = P(r_1 = -1) = \frac{1}{2}$ and the (r_j) are independent from both the $(\tilde{f}_j)$ and the $(\tilde{g}_j)$.

For $j \geq 0$, let $\tilde{f}_{n+j+1} = \tilde{f}_{n+j} + \tilde{f}_{n+j}r_{j+1}$ and $\tilde{g}_{n+j+1} = \tilde{g}_{n+j} - \tilde{f}_{n+j}r_{j+1}$. By this sequence of double or nothings we have that for $j \geq n$, $\|\tilde{f}_j\|_1 = \|\tilde{f}_n\|_1$, yet

$$\lim_{m \rightarrow \infty} \lambda P(\tilde{g}_m \geq \lambda) = \lambda P(\tilde{f}_n + \tilde{g}_n \geq \lambda) > \beta \|\tilde{f}\|_1$$

and since we are assuming a strict inequality, we can choose an $m > n$ that satisfies

$$\lambda P(\tilde{g}_m \geq \lambda) > \beta \|\tilde{f}\|_1.$$

This immediately implies the sharpness in (3.2). To show the sharpness in (3.3), it suffices to use $\tilde{f}$ and $\tilde{g}$ to construct a submartingale F with a ± 1 -transform G such that

$$(3.6) \quad \lambda P(\sup_{j \geq 0} G_j \geq \lambda) > \frac{3}{2} \beta \|F\|_1.$$

Let $\alpha = P(\sup_{0 \leq j \leq m} \tilde{g}_j \geq \lambda)$ so that $\alpha > 0$ and let $\delta = (4\|\tilde{f}\|_1 - \lambda\alpha)/(6 - 6\alpha)$ (in the case $\alpha = 1$, let $\delta = 0$). By (3.2), $\lambda\alpha \leq 4\|\tilde{f}\|_1$, hence $\delta \geq 0$.

Let s and t be independent random variables, independent from the $(\tilde{f}_j)$ such that $P(s = \lambda/6) = \alpha$ and $P(s = \delta) = 1 - \alpha$, while $P(t = -1) = 2/3$ and $P(t = 2) = 1/3$. Note that $\mathbf{E}s \leq 2\|\tilde{f}\|_1/3$.

Let $F_0 = -s$, $G_0 = s$, $F_1 = F_0 + tF_0$, and $G_1 = G_0 - tF_0$. We then have that $\|F_1\|_1 = \|F_0\|_1 = \mathbf{E}s$.

Let $F_2 = F_1 - F_1$ and $G_2 = G_1 - F_1$. Thus $F_2 = 0$ a.s. while $G_2 = 6s$ on the set $\{t = 2\}$ and $G_2 = 0$ on the set $\{t = -1\}$. We then have that

$$P(F_2 = 0, G_2 = \lambda) = \alpha/3, \quad P(F_2 = 0, G_2 = 6\delta) = (1 - \alpha)/3,$$

$$P(F_2 = 0, G_2 = 0) = 2/3.$$

Let $A = \{G_2 = 0\}$ and, for $j \geq 1$, let $F_{2+j} = \mathbf{1}_A \tilde{f}_j$ and $G_{2+j} = G_2 + \mathbf{1}_A \tilde{g}_j$. Then by the independence of t and the $(\tilde{f}_j)$, F is a submartingale, G is a ± 1 -transform of F , and for $j \geq 1$ we have that $\|F_{2+j}\|_1 = 2\|\tilde{f}_j\|_1/3$, while

$$\begin{aligned} P(\sup_{0 \leq j \leq m+2} G_j \geq \lambda) &= P(\sup_{0 \leq j \leq 2} G_j \geq \lambda) + \frac{2}{3} P(\sup_{0 < j \leq m} \tilde{g}_j \geq \lambda) \\ &\geq \frac{1}{3} \alpha + \frac{2}{3} P(\sup_{0 < j \leq m} \tilde{g}_j \geq \lambda) = P(\sup_{0 \leq j \leq m} \tilde{g}_j \geq \lambda), \end{aligned}$$

so that

$$\lambda P(\sup_{0 \leq j \leq m+2} G_j \geq \lambda) \geq \lambda P(\sup_{0 \leq j \leq m} \tilde{g}_j \geq \lambda) > \beta \|\tilde{f}\|_1 \geq \frac{3\beta}{2} \|F\|_1.$$

4. APPLICATIONS TO STOCHASTIC INTEGRALS

Theorem 4.1. *With $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)_{t \geq 0}$ as in Section 1, suppose X is an adapted right-continuous submartingale with left limits such that $\mathbf{E}X_0$ is finite and H is a predictable process with values in the closed unit ball of $\mathbb{R}^\nu$. Then with Y defined by $Y_t = H_0 X_0 + \int_{(0,t]} H_s dX_s$, we have that, for $\lambda > 0$,*

$$(4.1) \quad \lambda P(Y^* \geq \lambda) \leq 4 \sup_{t \geq 0} \mathbf{E}X_t^+ - 2\mathbf{E}X_0,$$

so that

$$(4.2) \quad \lambda P(Y^* \geq \lambda) \leq 6 \|X\|_1$$

and if $X_0 \equiv 0$, then

$$(4.3) \quad \lambda P(Y^* \geq \lambda) \leq 4 \|X\|_1.$$

The constants 6 and 4 in (4.2) and (4.3) respectively are the best possible.

Proof. As in Theorem 2.1 we have $\mathbf{E}X_t^+ \leq \|X_t\|_1 \leq 2\mathbf{E}X_t^+ - \mathbf{E}X_0$, hence we can assume the finiteness of $\|X\|_1$. The proof follows in the same way as the proof of Theorem 5.1 of [5], except that we use Theorem 2.1 above to show that X is an $L^{1,\infty}$ -integrator in the sense of [1].

The sharpness in (4.2) and (4.3) follow from the sharpness in (3.8) and (3.7) holding even for ± 1 -transforms.

5. LACK OF L^p INEQUALITIES

Fix $p \in [1, \infty)$ and $\beta > 1$. We shall construct a discrete time submartingale $F = (F_0, F_1, \dots)$ with $F_0 = 0$ and a ± 1 -transform of F , $G = (G_0, G_1, \dots)$ such that

$$(5.1) \quad \|G\|_1 > \beta \|F\|_p.$$

To do this, we will first construct a finite length submartingale $f = (f_0, f_1, \dots, f_N)$ with $f_0 = 0$, and $f_N \geq 0$ together with a ± 1 -transform of f , $g = (g_0, \dots, g_N)$ such that

$$\|g\|_1 > \beta \|f^+\|_p$$

where $\|f^+\|_p = \sup_{0 \leq n \leq N} \mathbf{E}(f_n^+)^p$. Let $M > 4\beta$ and let $(r_1, \dots, r_{2M})$ be a sequence of independent random variables such that for $j = 1, 2$, $P(r_j = 1) = P(r_j = -1) = 1/2$ and for $2 \leq j \leq M$,

$$P(r_{2j-1} = 1) = P(r_{2j-1} = -1) = \frac{1}{2},$$

$$P(r_{2j} = -1) = \frac{1}{3}, \quad P(r_{2j} = \frac{1}{2}) = \frac{2}{3}.$$

Let $f_j = \sum_{k=0}^j d_k$ where $d_0 = 0$, $d_1 = r_1/2$, $d_2 = r_2/2$, and $d_j = \mathbf{1}_{\{f_{j-1} < 0\}} r_j f_{j-1}$ for $j > 2$. By the independence of the r_j , $(f_j)_{j \leq 2M}$ forms a martingale. Note that for $j \geq 1$,

$$P(f_{2j} = 1) = \frac{1}{4}, \quad P(f_{2j} = -3^{j-1}) = \frac{1}{4} \left(\frac{1}{3}\right)^{j-1},$$

$$P(f_{2j} = 0) = \frac{3}{4} - \frac{1}{4} \left(\frac{1}{3}\right)^{j-1}.$$

For $0 \leq j \leq 2M$, let $g_j = \sum_{k=0}^j (-1)^k d_k$. Then for $2 \leq j \leq 2M$, $\|f_j^+\|_p^p = \|f_j^-\|_1 = 1/4$, while for $j \geq 1$,

$$\|g_{2j+2}\|_1 = \|g_{2j+1}\|_1 = \|g_{2j}\|_1 + 1/4.$$

Since $\|g_2\|_1 = 1/2$, it follows that $\|g_{2M}\|_1 = (M+1)/4$. Now let $N = 2M+1$, $f_N = f_{2M}^+$, and $g_N = g_{2M} + \mathbf{1}_{\{f_{2M} < 0\}} |f_{2M}|$. Then $f = (f_0, \dots, f_N)$ forms a submartingale, $\|f^+\|_p^p = 1/4$, and, since $f_{2M} < 0$ implies $g_{2M} = 0$, $\|g_N\|_1 \geq \|g_{2M}\|_1 - \|f_{2M}^-\|_1 = M/4 > \beta \|f^+\|_p$ by our choice of M .

To construct F and G , we will work with only a small portion of the probability space at a time in order to keep $\|F\|_p$ close to that of $\|f^+\|_p$. More explicitly, by enriching the probability space if necessary, let $A_1, \dots, A_K$ be a partition of the space such that $\sigma(A_1, \dots, A_K)$ is independent of $\sigma(f_0, \dots, f_N)$ and, for $1 \leq j \leq k$, $P(A_j) \leq \epsilon/3^{Mp}$, where ϵ satisfies $\beta^p(\|f^+\|_p^p + \epsilon) < \|g_N\|_1^p$.

Let $F_0 = G_0 = 0$ and for $1 \leq k \leq K$ and $1 \leq n \leq N$, let

$$F_{(k-1)N+n} = F_{(k-1)N} + \mathbf{1}_{A_k} f_n, \quad G_{(k-1)N+n} = G_{(k-1)N} + \mathbf{1}_{A_k} g_n.$$

Then F is a submartingale and G is a ± 1 -transform of F . Since $A_1, \dots, A_N$ partition the space, $G_{KN} = g_N$ and for $1 \leq k \leq K$ and $1 \leq n \leq N$, the disjointness of the A_j gives us

$$\|F_{(k-1)N+n}\|_p^p = \left\| f_N \mathbf{1} \left(\bigcup_{j=1}^{k-1} A_j \right) \right\|_p^p + \|f_n \mathbf{1}_{A_k}\|_p^p.$$

Since $f_N \geq 0$ a.s. and the f_j are bounded in absolute value by 3^M , we have that

$$\|F_{(k-1)N+n}\|_p^p \leq \|f_N^+\|_p^p + 3^{Mp} P(A_k) \leq \|f^+\|_p^p + \epsilon$$

which gives us (5.1) by our choice of ϵ .

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