CHAINS OF STRONGLY NON-REFLEXIVE DUAL GROUPS OF INTEGER-VALUED CONTINUOUS FUNCTIONS

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Dedicated to Professor Shōzō Sasada on his 60th birthday

ABSTRACT. Answering a question of Eklof-Mekler (*Almost free modules, set-theoretic methods*, North-Holland, Amsterdam, 1990), we prove: (1) If there exists a non-reflecting stationary set of ω_i consisting of ordinals of cofinality ω for each $1 < i < \omega$, then there exist abelian groups $A_n (n \in \mathbb{Z})$ such that $A_n^* \cong A_{n+1}$ and $A_n \not\cong A_{n+2}$ for each $n \in \mathbb{Z}$. (2) There exist abelian groups $A_n (n \in \mathbb{Z})$ such that $A_n^* \cong A_{n+1}$ for each $n \in \mathbb{Z}$ and $A_n \not\cong A_{n+2}$ for each $n < 0$. The groups A_n are the groups of $\mathbb{Z}$ -valued continuous functions on a topological space and their dual groups.

1. INTRODUCTION

For an abelian group A , let A^* denote the $\mathbb{Z}$ -dual group of A , i.e., $A^* = \text{Hom}(A, \mathbb{Z})$, where $\mathbb{Z}$ is the group of integers. An abelian group A is said to be *reflexive* if the canonical homomorphism $A \rightarrow A^{**}$ is an isomorphism, and A is said to be *strongly non-reflexive* if A is not isomorphic to A^{**} . In [4, p.455], Eklof and Mekler asked: Is there a $\mathbb{Z}$ -chain of strongly non-reflexive dual groups, i.e., abelian groups $A_n (n \in \mathbb{Z})$ such that for all n , $A_n^* = A_{n+1}$ and A_n is not isomorphic to A_{n+2} ? The purpose of this note is to give a consistency answer and a partial answer to this question by using the abelian groups $C(X, \mathbb{Z})$ of continuous functions from a topological space X to the discrete group $\mathbb{Z}$. For a regular, uncountable cardinal κ , let $E(\kappa)$ be the following statement: There exists a non-reflecting stationary set of κ consisting of ordinals of cofinality ω . Here, a subset E of κ is *non-reflecting* if $E \cap \lambda$ is not stationary in λ for each ordinal $\lambda < \kappa$ with cofinality $> \omega$. Obviously, $E(\omega_1)$ is true. It is known (cf. [4]) that under $V = L$, $E(\kappa)$ holds for any regular, not weakly compact cardinal κ . In section 3, we prove:

Theorem 1. *If $E(\omega_i)$ holds for each $1 < i < \omega$, then there exist topological spaces $X_n (n \in \mathbb{Z})$ such that, if $A_{2n} = C(X_n, \mathbb{Z})$ and $A_{2n+1} = A_{2n}^*$ for each $n \in \mathbb{Z}$, then $A_n^* \cong A_{n+1}$ and $A_n \not\cong A_{n+2}$ for each $n \in \mathbb{Z}$.*

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Theorem 2. *There exist topological spaces $X_n (n \in \mathbb{Z})$ such that, if $A_{2n} = C(X_n, \mathbb{Z})$ and $A_{2n+1} = A_{2n}^*$ for each $n \in \mathbb{Z}$, then $A_n^* \cong A_{n+1}$ for each $n \in \mathbb{Z}$ and $A_n \not\cong A_{n+2}$ for each $n < 0$.*

Strongly non-reflexive dual groups were constructed by several authors ([3], [4], [5] and [6]) and, in particular, Sageev-Shelah and Eklof-Mekler-Shelah [5] constructed an ω -chain of strongly non-reflexive dual groups assuming $\diamond_{\omega_1}$. The history of non-reflexive dual groups can be found in [4] and [5].

Throughout the paper, by a group we mean an abelian group and by a space we mean a 0-dimensional T_0 -space, i.e., a topological space with a base $\mathcal{B}$, consisting of clopen (= closed and open) sets, such that $\bigcap \{U : x \in U \in \mathcal{B}\} = \{x\}$ for each $x \in X$. Suppose that $\{X_\alpha\}_{\alpha \in A}$ is a collection of spaces. Take a copy X'_α of X_α for each α so that $X'_\alpha \cap X'_\beta = \emptyset$ if $\alpha \neq \beta$. Let $X = \bigcup_{\alpha \in A} X'_\alpha$ and define a topology for X by declaring a subset $U \subset X$ to be open if and only if $U \cap X'_\alpha$ is open in X'_α for each $\alpha \in A$. The space X with this topology is called the *discrete sum* of $\{X_\alpha\}_{\alpha \in A}$, and is denoted by $\bigsqcup_{\alpha \in A} X_\alpha$, or by $X_1 \sqcup X_2$ if $A = \{1, 2\}$. The cardinality of a set A is denoted by $|A|$. For ordinals λ and μ with $\lambda < \mu$, we write $[\lambda, \mu] = \{\alpha : \lambda < \alpha \leq \mu\}$ and $[\lambda, \mu) = \{\alpha : \lambda \leq \alpha < \mu\}$; as usual, we also write μ for $[0, \mu)$. Other terminology and notation will be used as in [4].

2. PRELIMINARIES

We state two lemmas, which play key roles in our proof of Theorems 1 and 2, together with some definitions. Except for Lemma 2 below, each definition and result stated here appears in [2] or [4]. A space X is called $\mathbb{Z}$ -compact if it is homeomorphic to a closed subspace of a product $\mathbb{Z}^\kappa$ for some cardinal κ . It is known that each space X can be densely embedded in a unique $\mathbb{Z}$ -compact space $\beta_{\mathbb{Z}}X$ to which each $\mathbb{Z}$ -valued continuous function on X admits a continuous extension. By the property of $\beta_{\mathbb{Z}}X$, the groups $C(X, \mathbb{Z})$ and $C(\beta_{\mathbb{Z}}X, \mathbb{Z})$ are isomorphic. A $\mathbb{Z}$ -valued function on a space X is said to be k -continuous if the restriction $f|_K$ is continuous for each compact set K of X . Let $k_{\mathbb{Z}}X$ be the set X equipped with the smallest topology so that every $\mathbb{Z}$ -valued k -continuous function on X is continuous.

Lemma 1 ([1, Lemma 4.4]). *If X is a $\mathbb{Z}$ -compact space and no compact set of X is of measurable cardinality, then $C(X, \mathbb{Z})^{**}$ is isomorphic to $C(k_{\mathbb{Z}}X, \mathbb{Z})$.*

For a space X , let $w(X) = \omega \cdot \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base of } X\}$.

Lemma 2 ([3, Theorem 3.1]). *Let X be a space and κ an infinite cardinal. Then there exists a subgroup of $C(X, \mathbb{Z})^*$ isomorphic to $\mathbb{Z}^\kappa$ if and only if there exists a compact set K of $\beta_{\mathbb{Z}}X$ with $w(K) \geq \kappa$.*

A *clopen ultrafilter* on a space X is a collection p of clopen sets in X which has the finite intersection property and has the property that for every clopen set U in X , exactly one of U and $X \setminus U$ is in p . A clopen ultrafilter p is ω_1 -complete if p intersects with each countable cover of X consisting of clopen sets, and p converges to a point x if $\bigcap \{V : V \in p\} = \{x\}$. A clopen ultrafilter which converges is also called principal. It is known (cf. [2] or [4]) that a space X is $\mathbb{Z}$ -compact if and only if every ω_1 -complete clopen ultrafilter on X converges. It follows that the discrete sum $\bigsqcup_{\alpha \in A} X_\alpha$ of $\mathbb{Z}$ -compact spaces X_α is $\mathbb{Z}$ -compact when $|A|$ is non-measurable.

The notion of $\mathbb{Z}$ -compact spaces originated with Mrówka (see the references of [2]); many topologists use the term ' N -compact' instead of ' $\mathbb{Z}$ -compact'.

3. PROOF OF THEOREMS 1 AND 2

Throughout this section, let κ be a regular, uncountable, non-measurable, successor cardinal and Ω the set of all ordinals less than κ which are successors or limits of cofinality ω . For a subset X of $\kappa + 1$, we call the subspace topology on X induced from the order topology the *relative order topology*. For a set X with $\kappa \in X \subset \kappa + 1$ and a topology τ on X , we call the space (X, τ) an *adequate space* if it satisfies the following conditions (1)-(4):

- (1) τ is finer than the relative order topology on X .
- (2) $\{(\lambda, \kappa] \cap X : \lambda < \kappa\}$ is a neighborhood base of the point κ in (X, τ) .
- (3) Each point of $X \cap \Omega$ has a countable neighborhood base in (X, τ) .
- (4) $\beta_{\mathbb{Z}}((X \cap \Omega) \cup \{\kappa\}) = (X, \tau)$, where $(X \cap \Omega) \cup \{\kappa\}$ is a subspace of (X, τ) .

Note that (4) implies that an adequate space is $\mathbb{Z}$ -compact. For a space Y , define $kw(Y) = \sup\{w(K) : \text{compact } K \subset Y\}$. For spaces X and Y , we write $X \approx Y$ if X is homeomorphic to Y .

Lemma 3. *Assume that $E(\kappa)$ holds and let E be a non-reflecting, stationary and co-stationary set of κ with $E \subset \Omega$. Let $X = (X, \tau)$ be an adequate space. Then there exists a topology v on $E \cup \{\kappa\}$ such that the following holds for the space $Y = (E \cup \{\kappa\}, v)$:*

- (a) Y is an adequate space.
- (b) $kw(Y) = \omega$.
- (c) $kw(\beta_{\mathbb{Z}}k_{\mathbb{Z}}Y) = kw(X)$.
- (d) $\beta_{\mathbb{Z}}k_{\mathbb{Z}}\beta_{\mathbb{Z}}k_{\mathbb{Z}}Y \approx \beta_{\mathbb{Z}}k_{\mathbb{Z}}Y \sqcup \beta_{\mathbb{Z}}k_{\mathbb{Z}}X$.

Proof. Since E can be decomposed into κ many disjoint stationary sets, we can write $E = \bigcup\{E_{\lambda} : \lambda \in (X \cap \Omega) \cup \{\kappa\}\}$, where each E_{λ} is stationary and $E_{\lambda} \cap E_{\mu} = \emptyset$ for $\lambda \neq \mu$. Moreover, we can assume that

$$(5) \quad \lambda \notin E_{\lambda} \text{ for each } \lambda \in X \cap \Omega.$$

Define a map $\varphi : E \rightarrow \Omega$ by $\varphi(\alpha) = \lambda$ if $\alpha \in E_{\lambda}$ for $\lambda \in X \cap \Omega$ and $\varphi(\alpha) = \alpha$ if $\alpha \in E_{\kappa}$. Put $G = \{\langle \alpha, \varphi(\alpha) \rangle : \alpha \in E\}$, $H = \{\langle \kappa, \lambda \rangle : \lambda \in X\}$ and $S = G \cup H (\subset (E \cup \{\kappa\}) \times (\kappa + 1))$. Topologize S as follows: A basic neighborhood of $\langle \alpha, \lambda \rangle \in S$ for $\alpha \notin E_{\kappa}$ is $((\beta, \alpha] \times U) \cap S$ for $\beta < \alpha$ and an open neighborhood U of λ in X such that $(\beta, \alpha] \cap U = \emptyset$, and a basic neighborhood of $\langle \alpha, \alpha \rangle \in S$ for $\alpha \in E_{\kappa}$ is $((\beta, \alpha] \times (\beta, \alpha]) \cap S$ for $\beta < \alpha$. By (1), (2) and (5), this defines a topology on S . Let T be the quotient space obtained from S by collapsing the set H to the point $\langle \kappa, \kappa \rangle$ and $\psi : S \rightarrow T$ the quotient map. Then there is a bijection $h : T \rightarrow E \cup \{\kappa\}$ such that $h \circ \psi = \pi$, where $\pi : S \rightarrow E \cup \{\kappa\}$ is the projection. Define v to be the topology on $E \cup \{\kappa\}$ which makes the map h a homeomorphism and put $Y = (E \cup \{\kappa\}, v)$.

To prove (a)-(d), let γ denote the relative order topology on $E \cup \{\kappa\}$. We begin by proving (a). By the definition, $\gamma \subset v$, and hence Y satisfies (1). By using (2) for X and the regularity of κ , it is easily checked that every open set in S including H includes an open set of the form $((\mu, \kappa] \times (\kappa + 1)) \cap S$ for some $\mu < \kappa$. This implies that Y satisfies (2). By the fact that $E \subset \Omega$ and (3) for X , each point of G has a countable neighborhood base. Hence, Y satisfies (3). Since $E \subset \Omega$, $Y = (Y \cap \Omega) \cup \{\kappa\}$. Thus, if we show that Y is $\mathbb{Z}$ -compact, then (4) would be satisfied. To do this and to prove Claim 4 below, let $Y_{\mu} = Y \cap \mu$ for each $\mu < \kappa$.

Claim 1. For each $\mu < \kappa$, Y_{μ} is $\mathbb{Z}$ -compact.

Proof. In case $\kappa = \omega_1$, this is obvious since each Y_μ is then countable. So we prove the case $\kappa > \omega_1$ by induction. Assume that $\mu < \kappa$ and Y_λ is $\mathbb{Z}$ -compact for each $\lambda < \mu$. Since a union of a $\mathbb{Z}$ -compact space and a singleton is $\mathbb{Z}$ -compact, Y_μ is $\mathbb{Z}$ -compact if μ is a successor. If μ is a limit, then either cofinality $\mu = \omega$ or Y_μ is not stationary in μ , because E is non-reflecting. In each case, since $\gamma \subset v$, we can write $Y_\mu = \bigsqcup_{\xi \in \Xi} A_\xi$, where each A_ξ is bounded in μ and clopen in Y . By our assumption, each A_ξ is $\mathbb{Z}$ -compact. Since κ is non-measurable, Y_μ is $\mathbb{Z}$ -compact. $\square$

To prove that Y is $\mathbb{Z}$ -compact, let p be an ω_1 -complete clopen ultrafilter on Y . If p does not converge to κ , then $\kappa \notin U$ for some $U \in p$. Since Y satisfies (2), there is $\mu < \kappa$ such that $U \subset Y_\mu$. By Claim 1, p converges to a point of U . Thus Y is $\mathbb{Z}$ -compact and hence satisfies (4).

Claim 2. Every compact set of Y is countable.

Proof. Suppose that $K \subset Y$ is compact and $|K| > \omega$. Then there is a strictly increasing sequence $\{\lambda_\alpha : \alpha < \omega_1\}$ in K . Put $\lambda = \sup \lambda_\alpha$. Since $\gamma \subset v$, K is compact in γ , and hence, $K \cap \lambda$ is closed and unbounded in λ with respect to its relative order topology. If $\kappa = \omega_1$, then $\lambda = \omega_1$ and this contradicts the fact that E is co-stationary. If $\kappa > \omega_1$, this contradicts the fact that E is non-reflecting. $\square$

The statement (b) follows from Claim 2 and (2), (3) for Y .

Claim 3. $k_{\mathbb{Z}}Y \approx G$ and $k_{\mathbb{Z}}S \approx G \sqcup k_{\mathbb{Z}}X$.

Proof. By Claim 2 and (2) for Y , the characteristic function of E on Y is k -continuous, which implies that $k_{\mathbb{Z}}Y \approx k_{\mathbb{Z}}E \sqcup \{\kappa\}$. Since each point of E has a countable neighborhood base, $k_{\mathbb{Z}}E = E$. Since $E \sqcup \{\kappa\} \approx E \approx G$, $k_{\mathbb{Z}}Y \approx G$. Next, let K be a compact set of S . Since the projection $\pi : S \rightarrow Y$ is continuous, it follows from Claim 2 that $|\pi[K]| \leq \omega$. This implies that the characteristic function of G on S is k -continuous. Hence, $k_{\mathbb{Z}}S \approx k_{\mathbb{Z}}G \sqcup k_{\mathbb{Z}}H$. Since $k_{\mathbb{Z}}G = G$ and $H \approx X$, $k_{\mathbb{Z}}S \approx G \sqcup k_{\mathbb{Z}}X$. $\square$

Claim 4. $\beta_{\mathbb{Z}}G = S$.

Proof. First, we show that S is $\mathbb{Z}$ -compact. Let p be an ω_1 -complete clopen ultrafilter on S . If there is $U \in p$ with $U \cap H = \emptyset$, then $U \subset \pi^{-1}[Y_\mu]$ for some $\mu < \kappa$. Since $\pi^{-1}[Y_\mu] \approx Y_\mu$, p converges by Claim 1. If $U \cap H \neq \emptyset$ for each $U \in p$, put $q = \{U \cap H : U \in p\}$. Observe that (i) for each clopen set V in H , there is a clopen set W in S such that $W \cap H = V$ and that (ii) if U is a clopen set in S with $U \cap H \neq \emptyset$, then there is $\nu < \kappa$ such that $((\nu, \kappa] \times \{\lambda \in X : \langle \kappa, \lambda \rangle \in U\}) \cap S \subset U$. (i) implies that q is a clopen ultrafilter on H and (ii) implies that q is ω_1 -complete. Since $H \approx X$ and X is $\mathbb{Z}$ -compact, q converges, and hence, p converges to the same point. Thus, S is $\mathbb{Z}$ -compact. Next, we show that each $\mathbb{Z}$ -valued continuous function f on G can be continuously extended over S . For each $\lambda \in (X \cap \Omega) \cup \{\kappa\}$, put $F_\lambda = \{\langle \alpha, \varphi(\alpha) \rangle : \alpha \in E_\lambda\}$. Then the subspace F_λ of G is homeomorphic to E_λ with the relative order topology. Since E_λ is stationary, there is $\beta(\lambda) < \kappa$ such that f takes a constant value $g(\lambda)$ on $\{\langle \alpha, \varphi(\alpha) \rangle \in F_\lambda : \alpha > \beta(\lambda)\}$. We show that g is a continuous function on the subspace $(X \cap \Omega) \cup \{\kappa\}$ of X . Applying Forder's lemma to F_κ , we can find $\nu_0 < \kappa$ such that $f[(\nu_0, \kappa] \times (\nu_0, \kappa] \cap G] = \{g(\kappa)\}$. Thus, $g(\xi) = g(\kappa)$ for each $\xi \in X \cap \Omega \cap (\nu_0, \kappa]$, which shows that g is continuous at κ . For $\lambda \in X \cap \Omega$, put $\beta_0 = \sup_{\xi \leq \lambda} \beta(\xi)$ and $F = \bigcap_{\xi < \lambda} \text{cl} E_\xi$, where the closure is

taken in κ with the order topology. Then F is closed and unbounded, and hence, there is $\beta_1 \in F \cap E_\lambda \cap (\beta_0, \kappa]$. Since $f(\langle \beta_1, \lambda \rangle) = g(\lambda)$, there is a neighborhood U of λ in X such that $f[\langle (\beta_0, \beta_1] \times U \cap G] = \{g(\lambda)\}$. By the condition (1) for X , we may assume that $U \subset \lambda + 1$. Since $(\beta_0, \beta_1] \cap E_\xi \neq \emptyset$ for each $\xi \in U \cap \Omega$, $g[U \cap \Omega] = \{g(\lambda)\}$. Hence, g is continuous at λ . By the condition (4) for X , g can be extended to a continuous function $\bar{g}$ on X . Extend f over S by letting $f(\langle \kappa, \lambda \rangle) = \bar{g}(\lambda)$ for each $\lambda \in X$. Then, it is easily checked that the extension is continuous. Hence, $\beta_{\mathbb{Z}}G \approx S$. $\square$

By Claims 3 and 4, $\beta_{\mathbb{Z}}k_{\mathbb{Z}}Y \approx \beta_{\mathbb{Z}}G \approx S$. To prove (c), let K be a compact set of S . Then, by Claim 2, $|\pi[K]| \leq \omega$, which implies that $K \cap G$ is countable and clopen in K . Since each point of G has a countable neighborhood base, $w(K \cap G) \leq \omega$. Since $H \approx X$, $w(K) \leq kw(X)$, and hence, $kw(S) \leq kw(X)$. Conversely, since $X \approx H \subset S$, $kw(X) \leq kw(S)$. Thus, we have (c). Finally, by Claim 3, $\beta_{\mathbb{Z}}k_{\mathbb{Z}}S \approx \beta_{\mathbb{Z}}G \sqcup \beta_{\mathbb{Z}}k_{\mathbb{Z}}X \approx \beta_{\mathbb{Z}}k_{\mathbb{Z}}Y \sqcup \beta_{\mathbb{Z}}k_{\mathbb{Z}}X$. Since $\beta_{\mathbb{Z}}k_{\mathbb{Z}}Y \approx S$, we have (d), which completes the proof of Lemma 3. $\square$

Lemma 4. Assume that $E(\kappa)$ holds. Then there exists a collection $\{Y_{\kappa,n} : n \in \mathbb{Z}\}$ of $\mathbb{Z}$ -compact spaces such that:

- (a') $\beta_{\mathbb{Z}}k_{\mathbb{Z}}Y_{\kappa,n} \approx Y_{\kappa,n+1}$ for each $n \in \mathbb{Z}$,
- (b') $Y_{\kappa,n} = Y_{\kappa,1}$ for each $n \geq 1$,
- (c') $kw(Y_{\kappa,n}) = \omega$ for each $n < 0$, and
- (d') $kw(Y_{\kappa,n}) = \kappa$ for each $n \geq 0$.

Before proving Lemma 4, we define some symbols. For a space X , let $\bigsqcup_{\omega} X$ denote the discrete sum of ω many copies of X ; in other words, $\bigsqcup_{\omega} X = X \times \mathbb{Z}$. We denote the space Y constructed from an adequate space X in Lemma 3 by $\Delta(X)$. Since $\Delta(X)$ is an adequate space again, we can define $\Delta(\Delta(X))$. Inductively, we write $\Delta^{i+1}(X) = \Delta(\Delta^i(X))$ for each $i \geq 0$, where $\Delta^0(X) = X$.

Proof of Lemma 4. Let $X = \kappa + 1$ with the order topology. Then X is compact, and hence, $\beta_{\mathbb{Z}}k_{\mathbb{Z}}X = X$ and $kw(X) = \kappa$. Since X is an adequate space, we can define the space $\Delta^i(X)$ for each $i \geq 0$. For each $n \in \mathbb{Z}$, define

$$Y_{\kappa,n} = \bigsqcup_{i \geq m(n)} \left(\bigsqcup_{\omega} \beta_{\mathbb{Z}}k_{\mathbb{Z}}\Delta^i(X) \right), \text{ where } m(n) = \max\{1 - n, 0\}.$$

Then $Y_{\kappa,n}$ is $\mathbb{Z}$ -compact and $Y_{\kappa,n} = Y_{\kappa,1}$ for each $n \geq 1$; thus (b') holds. For $n \geq 1$, by (d) in Lemma 3,

$$\begin{aligned} \beta_{\mathbb{Z}}k_{\mathbb{Z}}Y_{\kappa,n} &\approx \bigsqcup_{\omega} \beta_{\mathbb{Z}}k_{\mathbb{Z}}\beta_{\mathbb{Z}}k_{\mathbb{Z}}\Delta^0(X) \\ &\sqcup \left(\bigsqcup_{i \geq 1} \left(\bigsqcup_{\omega} (\beta_{\mathbb{Z}}k_{\mathbb{Z}}\Delta^i(X) \sqcup \beta_{\mathbb{Z}}k_{\mathbb{Z}}\Delta^{i-1}(X)) \right) \right). \end{aligned}$$

Since $\beta_{\mathbb{Z}}k_{\mathbb{Z}}\Delta^0(X) = \Delta^0(X)$, $\beta_{\mathbb{Z}}k_{\mathbb{Z}}Y_{\kappa,n}$ is homeomorphic to $Y_{\kappa,n+1}$. For $n \leq 0$,

$m(n) - 1 = m(n + 1)$ and by (d) in Lemma 3 again,

$$\begin{aligned}\beta_{\mathbb{Z}}k_{\mathbb{Z}}Y_{\kappa,n} &\approx \bigsqcup_{i \geq m(n)} \left(\bigsqcup_{\omega} (\beta_{\mathbb{Z}}k_{\mathbb{Z}}\Delta^i(X) \sqcup \beta_{\mathbb{Z}}k_{\mathbb{Z}}\Delta^{i-1}(X)) \right) \\ &\approx \bigsqcup_{i \geq m(n+1)} \left(\bigsqcup_{\omega} \beta_{\mathbb{Z}}k_{\mathbb{Z}}\Delta^i(X) \right) \approx Y_{\kappa,n+1}.\end{aligned}$$

Hence, we have (a'). By (b) and (c) in Lemma 3, $kw(\beta_{\mathbb{Z}}k_{\mathbb{Z}}\Delta^i(X)) = kw(\Delta^{i-1}(X)) = \omega$ for each $i \geq 2$ and $kw(\beta_{\mathbb{Z}}k_{\mathbb{Z}}\Delta^i(X)) = kw(X) = \kappa$ for $i = 0, 1$. These imply (c') and (d'). $\square$

Proof of Theorem 1. Assume that $E(\omega_i)$ holds for each $1 < i < \omega$. Note that $E(\omega_1)$ is always true. By Lemma 4, for each $i \geq 1$, there exists a collection $\{Y_{\omega_i,n} : n \in \mathbb{Z}\}$ of $\mathbb{Z}$ -compact spaces satisfying (a')-(d'). For each $n \in \mathbb{Z}$, define $X_n = \bigsqcup_{i \geq 1} Y_{\omega_i,n-i}$. Then X_n is $\mathbb{Z}$ -compact and by (a'),

$$(6) \quad \beta_{\mathbb{Z}}k_{\mathbb{Z}}X_n \approx X_{n+1} \text{ for each } n \in \mathbb{Z}.$$

Moreover, by (c') and (d'),

$$(7) \quad kw(X_n) = \omega_n \text{ for each } n \geq 0,$$

where $\omega_0 = \omega$. Put $A_{2n} = C(X_n, \mathbb{Z})$ and $A_{2n+1} = A_{2n}^*$ for each $n \in \mathbb{Z}$. Then, it follows from (6) and Lemma 1 that

$$\begin{aligned}A_{2n+1}^* &= C(X_n, \mathbb{Z})^{**} \\ &\cong C(k_{\mathbb{Z}}X_n, \mathbb{Z}) \\ &\cong C(\beta_{\mathbb{Z}}k_{\mathbb{Z}}X_n, \mathbb{Z}) \\ &\cong C(X_{n+1}, \mathbb{Z}) = A_{2n+2}.\end{aligned}$$

If $n \geq 0$, then by (7) and Lemma 2, no subgroup of A_{2n+1} is isomorphic to $\mathbb{Z}^{\omega_{n+1}}$ but A_{2n+3} has a subgroup isomorphic to $\mathbb{Z}^{\omega_{n+1}}$. Hence, $A_{2n+1} \not\cong A_{2n+3}$ for $n \geq 0$. This means that A_n is strongly non-reflexive for each $n \in \mathbb{Z}$. $\square$

Proof of Theorem 2. Since $E(\omega_1)$ is always true, there exists a collection $\{Y_{\omega_1,n} : n \in \mathbb{Z}\}$ of $\mathbb{Z}$ -compact spaces satisfying (a')-(d') by Lemma 4. For each $n \in \mathbb{Z}$, define $X_n = Y_{\omega_1,n}$, and put $A_{2n} = C(X_n, \mathbb{Z})$ and $A_{2n+1} = A_{2n}^*$. Then, similarly to the proof of Theorem 1, $A_n^* \cong A_{n+1}$ for each $n \in \mathbb{Z}$. By (c'), $kw(X_{-1}) = \omega$ and by (d'), $kw(X_0) = \omega_1$. Hence, it follows from Lemma 2 that $A_{-1} \not\cong A_1$. This means that A_n is strongly non-reflexive for each $n < 0$. $\square$

Remark 1. Let X_n and A_n ($n \in \mathbb{Z}$) be the same as in the proof of Theorem 2. By the proof of Lemma 3, $k_{\mathbb{Z}}X_n \neq X_n$ for each $n \in \mathbb{Z}$. Hence, it follows from [1, Theorem 4.1] that A_n is non-reflexive for all $n \geq 0$ too. However, by (b') in Lemma 4, $A_n \cong A_{n+2}$ for each $n \geq 2$. With some effort, it can also be proved that $A_n \cong A_{n+2}$ for $n = 0, 1$.

Remark 2. Our construction yields another strongly non-reflexive dual group $A = C(Y, \mathbb{Z})$. Let $X = \{\omega_1\} \cup \{\alpha < \omega_1 : \alpha \text{ is a successor}\}$ with the relative order

topology. Then X is an adequate space. Define $Y = \beta_{\mathbb{Z}} k_{\mathbb{Z}} \Delta(X)$ and $A = C(Y, \mathbb{Z})$; then A is a dual group by Lemma 1. Since $k_{\mathbb{Z}} \Delta(X)$ has no partition by ω_1 many non-empty clopen sets, it follows from [3, Theorem 3.3] that $A (\cong C(k_{\mathbb{Z}} \Delta(X), \mathbb{Z}))$ has no subgroup isomorphic to $\mathbb{Z}^{\omega_1}$. On the other hand, by Lemma 1, $A^{**} \cong C(\beta_{\mathbb{Z}} k_{\mathbb{Z}} Y, \mathbb{Z})$ and by (d) in Lemma 3, $\beta_{\mathbb{Z}} k_{\mathbb{Z}} Y \approx Y \sqcup \beta_{\mathbb{Z}} k_{\mathbb{Z}} X$. Since every compact set of X is finite, $k_{\mathbb{Z}} X$ is a discrete space, and hence, so is $\beta_{\mathbb{Z}} k_{\mathbb{Z}} X$. These mean that there is a summand of A^{**} isomorphic to $\mathbb{Z}^{\omega_1}$. Hence, A is strongly non-reflexive. Assuming $E(\kappa)$, ω_1 can be replaced by κ .

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