RIGHT ADJOINT FOR THE SMASH PRODUCT FUNCTOR

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ABSTRACT. The smash-product functor $(-) \wedge (X, x_0)$ in the category $\mathbf{Top}_*$ of pointed topological spaces has a right adjoint for any choice of the base point x_0 , if and only if the topological space X is quasi-locally compact, that is, if and only if the product functor $(-) \times X$ has a right adjoint in the category $\mathbf{Top}$ of topological spaces.

INTRODUCTION

A space X is cartesian in the category of topological spaces and continuous maps if the product functor $(-) \times X$ has a right adjoint. This means that there exists a proper and admissible topology on the space of maps Y^X between X and Y (for any topological space Y) [D]. Cartesian objects in $\mathbf{Top}$ were characterized by Day and Kelly [D-K]. They are the quasi-locally compact spaces [H-L].

The problem of the existence of a proper and admissible topology on the function space $(Y, y_0)^{(X, x_0)}$ consisting of the maps of Y^X preserving base points is related to the adjointness of the smash-product functor. It is known that this functor has a right adjoint whenever X is locally compact and Hausdorff; in this case the topology on $(Y, y_0)^{(X, x_0)}$ is the compact open topology [M].

In this paper, it is proved that the spaces (X, x_0) for which the functor $(-) \wedge (X, x_0)$ has a right adjoint are exactly the spaces X which are cartesian in $\mathbf{Top}$, independently of the choice of x_0 . That is, the existence of a proper and admissible topology on $(Y, y_0)^{(X, x_0)}$ for any (Y, y_0) is equivalent to the existence of a proper and admissible topology on the whole space of maps from X to Y , for any Y .

SMASH-PRODUCT AND ADJUNCTION

We can consider, in $\mathbf{Top}_*$, the endofunctor $(-) \wedge (X, x_0)$ and ask when it has a right adjoint. When it exists we will call it $\mathcal{G}_{(X, x_0)}$.

In the case of X cartesian in $\mathbf{Top}$, we indicate by Y^X the power object and by $(Y, y_0)^{(X, x_0)}$ the subspace of Y^X given by $\{f \in Y^X \mid f(x_0) = y_0\}$ with base point the constant y_0 -valued map.

Theorem 1. *If X is cartesian in $\mathbf{Top}$, then $(-) \wedge (X, x_0)$ has a right adjoint, for every x_0 in X . Moreover $\mathcal{G}_{(X, x_0)}(Y, y_0) = (Y, y_0)^{(X, x_0)}$.*

Proof. Suppose X is cartesian in **Top** and take any space Y ; let $\hat{e}: Y^X \times X \rightarrow Y$ be the evaluation map. Consider in Y^X the subspace $\mathcal{G}_{(X,x_0)}(Y, y_0) = \{f \in Y^X \mid f(x_0) = y_0\}$ and the restriction e_1 of $\hat{e}$ to $\mathcal{G}_{(X,x_0)}(Y, y_0) \times (X, x_0)$ which is a map in **Top**. The map e_1 is compatible with the quotient in the definition of the smash-product, and so we can consider the map $e: \mathcal{G}_{(X,x_0)}(Y, y_0) \wedge (X, x_0) \rightarrow (Y, y_0)$ induced by e_1 . We note that $e(f, x) = f(x)$.

Let $f: (Z, z_0) \wedge (X, x_0) \rightarrow (Y, y_0)$ be a map in **Top** and consider the quotient $p: (Z, z_0) \times (X, x_0) \rightarrow (Z, z_0) \wedge (X, x_0)$ which gives the smash-product. Since X is cartesian in **Top**, related to $fp: (Z, z_0) \times (X, x_0) \rightarrow (Y, y_0)$ there is an $(fp)_1: Z \rightarrow Y^X$ such that $\hat{e}((fp)_1 \times \text{id}_X) = fp$. The map $(fp)_1$ preserves the base points and its image is a subspace of $\mathcal{G}_{(X,x_0)}(Y, y_0)$, so we can factor $(fp)_1$ through the inclusion of $\mathcal{G}_{(X,x_0)}(Y, y_0)$ in Y^X and consider the first factor $(fp)_2$ as a map in **Top**. In such a way we obtain $(fp)_2 \times \text{id}_X: (Z, z_0) \times (X, x_0) \rightarrow (Y, y_0)$. By construction $(fp)_2 \times \text{id}_X$ is compatible with the quotient p , and the proof is complete.

Theorem 2. *If the functor $(-) \wedge (X, x_0)$ has a right adjoint, then $\mathcal{G}_{(X,x_0)}(Y, y_0)$ is a space whose underlying set is in natural bijective correspondence with $(Y, y_0)^{(X,x_0)}$, the counit of the adjunction is the map $e: \mathcal{G}_{(X,x_0)}(Y, y_0) \wedge (X, x_0) \rightarrow (Y, y_0)$ such that $e(f, x) = f(x)$ and the base point corresponds to the constant function valued at y_0 .*

Proof. Let D_2 be the space with two points a, b and the discrete topology. By the adjunction, there is a bijection between $(Y, y_0)^{(D_2, a) \wedge (X, x_0)}$ and $(\mathcal{G}_{(X,x_0)}(Y, y_0))^{(D_2, a)}$, and on the other side $(D_2, a) \wedge (X, x_0)$ is homeomorphic to (X, x_0) and $(\mathcal{G}_{(X,x_0)}(Y, y_0))^{(D_2, a)}$ is in bijection with $\mathcal{G}_{(X,x_0)}(Y, y_0)$; so the first part of the theorem is proved.

Any map $f: (X, x_0) \rightarrow (Y, y_0)$ can be considered as a map from $(D_2, a) \wedge (X, x_0)$ into (Y, y_0) . As a consequence, by the adjunction, for any f , there is an $f_1: (D_2, a) \rightarrow \mathcal{G}_{(X,x_0)}(Y, y_0)$ such that $e(f_1 \wedge \text{id}_X) = f$; so $e(f, x) = f(x)$. Finally, given the one point space $\bullet$, and the map $h: \bullet \wedge (X, x_0) \rightarrow (Y, y_0)$, there is a map $h_1: \bullet \rightarrow \mathcal{G}_{(X,x_0)}(Y, y_0)$ such that $h_1(\bullet)$ is the base point of $\mathcal{G}_{(X,x_0)}(Y, y_0)$. This completes the proof.

We denote by S the Sierpinski space with the two points 0 and 1 and $\{0\}$ the nontrivial open set. If the functor $(-) \wedge (X, x_0)$ has a right adjoint, as a consequence of Theorem 2, $\mathcal{G}_{(X,x_0)}(S, 0)$ can be identified with the set of the open sets U of X such that $x_0 \in U$ and base point the open set X . On the other hand, $\mathcal{G}_{(X,x_0)}(S, 1)$ can be identified with the set of the open sets U of X such that $x_0 \notin U$ and base point the empty set.

The following Lemma characterizes convergent nets of the spaces $\mathcal{G}_{(X,x_0)}(S, 0)$ (respectively, $\mathcal{G}_{(X,x_0)}(S, 1)$), while Lemma 4 proves that the open sets of these spaces are Scott-open [H-L].

Lemma 3. *Suppose $(-) \wedge (X, x_0)$ admits a right adjoint. A net U_i converges to U in $\mathcal{G}_{(X,x_0)}(S, 0)$ (respectively, $\mathcal{G}_{(X,x_0)}(S, 1)$) if and only if:*

- (*) *for each $x \in U$ and for each net x_λ converging to x in X , there is an i' and a λ' such that $x_\lambda \in U_i$, for every $i > i'$ and $\lambda > \lambda'$.*

Proof. Let U_i converge to U in $\mathcal{G}_{(X,x_0)}(S, 0)$ (respectively, $\mathcal{G}_{(X,x_0)}(S, 1)$), $x \in U$ and x_λ converge to x in X . Consider the counit of the adjunction $e: \mathcal{G}_{(X,x_0)}(S, 0) \wedge$

$(X, x_0) \rightarrow (S, 0)$ (respectively, $(S, 1)$) and the quotient map $p: \mathcal{G}_{(X, x_0)}(S, 0) \times (X, x_0) \rightarrow \mathcal{G}_{(X, x_0)}(S, 0) \wedge (X, x_0)$ (respectively, $(S, 1)$). Since (U_i, x_λ) converges to (U, x) and the map ep is continuous with $ep(U, x) = 0$, then $ep(U_i, x_\lambda)$ converges to 0. In S the nets converging to 0 are eventually constant, so there is an i' and a λ' such that, for every $i > i'$ and $\lambda > \lambda'$, $ep(U_i, x_\lambda) = 0$, that is $x_\lambda \in U_i$.

Vice versa, suppose U_i is a net in $\mathcal{G}_{(X, x_0)}(S, 0)$ and $U \in \mathcal{G}_{(X, x_0)}(S, 0)$ (respectively, $\mathcal{G}_{(X, x_0)}(S, 1)$) fulfilling condition (*). Consider the space $I \cup \{\bullet\}$, where I is the direct set of the net U_i , $\bullet$ is a maximum point whose base-neighborhoods are the sets of the form $I_j = \{\bullet\} \cup \{i \in I \mid i \geq j\}$, $j \in I$, and the points of I are isolated. We can assume, without changing the nature of the net U_i , that there exists a point $i_0 \in I$, such that $U_{i_0} = X$ (respectively, $U_{i_0} = \emptyset$). Moreover, we consider the map $\alpha: (I \cup \{\bullet\}, i_0) \rightarrow \mathcal{G}_{(X, x_0)}(S, 0)$ (respectively, $\mathcal{G}_{(X, x_0)}(S, 1)$) so defined by $\alpha(i) = U_i$, $\alpha(\bullet) = U$. We prove that the map α is continuous, which implies the convergence of the net U_i to U . By the existence of the right adjoint, α is continuous if and only if the corresponding map $e(\alpha \wedge \text{Id}_X) = \bar{\alpha}: (I \cup \{\bullet\}, i_0) \wedge (X, x_0) \rightarrow (S, 0)$ (respectively, $(S, 1)$) is continuous; so we will prove the continuity of $\bar{\alpha}$. To this aim we can consider the quotient map $q: (I \cup \{\bullet\}, i_0) \times (X, x_0) \rightarrow (I \cup \{\bullet\}, i_0) \wedge (X, x_0)$ induced by the smash-product and prove the continuity of $\bar{\alpha}q$. By the adjunction $(\bar{\alpha}q)^{-1}(0) = \{(i, x) \mid x \in U_i, i \in I\} \cup \{(\bullet, x) \mid x \in U\}$. Since every point of I is isolated and each U_i is open, the set $\{(i, x) \mid x \in U_i, i \in I\}$ is open since it is union of open sets, therefore each (i, x) of $(\bar{\alpha}q)^{-1}(0)$ belongs to its interior. Take now $(\bullet, x) \in (\bar{\alpha}q)^{-1}(0)$ (that is $x \in U$); the topology defined on $(I \cup \{\bullet\})$ and condition (*) implies that any net converging to $(\bullet, x)$ in the product space $(I \cup \{\bullet\}, i_0) \times (X, x_0)$ is eventually in $(\bar{\alpha}q)^{-1}(0)$, that is, $(\bullet, x)$ belongs to the interior of $(\bar{\alpha}q)^{-1}(0)$. We can conclude that $(\bar{\alpha}q)^{-1}(0)$ is open and so, α is continuous.

Lemma 4. Suppose $(-) \wedge (X, x_0)$ admits a right adjoint, and suppose H is open in $\mathcal{G}_{(X, x_0)}(S, 0)$ (respectively, $\mathcal{G}_{(X, x_0)}(S, 1)$); then H is Scott-open [H-L], i.e.:

- (a) If U, U' are open in $X, U \in H$ and $U' \supseteq U$ (with $x_0 \notin U'$ when H is open in $\mathcal{G}_{(X, x_0)}(S, 1)$), then $U' \in H$.
- (b) If $V = \{U_i \mid i \in I\}$ is a family of open subsets of X and $U = \bigcup \{U_i \mid i \in I\} \in H$, there exists a finite subfamily of V whose union belongs to H .

Proof. (a) Suppose U, U' open in $X, U \in H$ and $U' \supseteq U$ ($x_0 \notin U'$ in the case when H is open in $\mathcal{G}_{(X, x_0)}(S, 1)$). The constant sequence whose value is U' converges to U in $\mathcal{G}_{(X, x_0)}(S, 0)$ (respectively, $\mathcal{G}_{(X, x_0)}(S, 1)$) because it fulfils the condition (*) in Lemma 3. Since H is open and U belongs to H , there must be an element of the constant sequence belonging to H ; consequently $U' \in H$.

(b) Let $U = \bigcup \{U_i \mid i \in I\} \in H$. If H is open in $\mathcal{G}_{(X, x_0)}(S, 0)$, it follows that $x_0 \in U$; therefore there exists an i' such that $x_0 \in U_{i'}$. Consider in $\mathcal{G}_{(X, x_0)}(S, 0)$ the net whose direct set is $\{(i_1, i_2, \dots, i_n) \mid i_1, i_2, \dots, i_n \in I\}$ with the relation $(i_1, i_2, \dots, i_n) \succ (j_1, j_2, \dots, j_m)$ if $U_{i_1} \cup U_{i_2} \cup \dots \cup U_{i_n} \supseteq U_{j_1} \cup U_{j_2} \cup \dots \cup U_{j_m}$ and image of $(i_1, i_2, \dots, i_n)$ equal to $U_{i'} \cup U_{i_1} \cup U_{i_2} \cup \dots \cup U_{i_n}$. This net, according to (*), converges to U in $\mathcal{G}_{(X, x_0)}(S, 0)$; as a consequence, since H is an open set which contains the limit of the net, there exists $(i_1, i_2, \dots, i_n)$ such that $U_{i'} \cup U_{i_1} \cup U_{i_2} \cup \dots \cup U_{i_n} \in H$. When H is open in $\mathcal{G}_{(X, x_0)}(S, 1)$, $x_0 \notin U$ and therefore, $x_0 \notin U_i$, for each $i \in I$. In this case, we can consider the net with the same direct set as above and with the image of $(i_1, i_2, \dots, i_n)$ equal to $U_{i_1} \cup U_{i_2} \cup \dots \cup U_{i_n}$. The same argument of the first case proves that there is $(i_1, i_2, \dots, i_n)$ such that $U_{i_1} \cup U_{i_2} \cup \dots \cup U_{i_n} \in H$.

Definition 5. A space X is *quasi-locally compact* if for every x in X and for every neighbourhood U of x there is a neighbourhood of $V \subseteq U$ of x such that every open cover of U has a finite subcover of V .

Theorem 6. Let X be a topological space. The following statements are equivalent:

- (1) There exists x_0 in X such that the functor $(-) \wedge (X, x_0): \mathbf{Top}_* \rightarrow \mathbf{Top}_*$ has a right adjoint.
- (2) X is cartesian in $\mathbf{Top}$, that is X is quasi-locally compact.
- (3) For any x_0 in X , the functor $(-) \wedge (X, x_0): \mathbf{Top}_* \rightarrow \mathbf{Top}_*$ has a right adjoint.

Proof. (1) $\rightarrow$ (2) Since the continuity of the counit of the adjunction $e: \mathcal{G}_{(X, x_0)}(S, 0) \wedge (X, x_0) \rightarrow (S, 0)$ implies the continuity of the evaluation map $e': \mathcal{G}_{(X, x_0)}(S, 0) \times (X, x_0) \rightarrow (S, 0)$, it follows that $(e')^{-1}(0)$ is an open set with $e'(W, x_0) = 0$, for any $W \in \mathcal{G}_{(X, x_0)}(S, 0)$.

Take a point $x \in X$ and fix U open in X with $x \in U$.

Suppose $x \in \text{cl}\{x_0\}$; then $x_0 \in U$. Consequently U belongs to $\mathcal{G}_{(X, x_0)}(S, 0)$ and $e'(U, x) = 0$. Since $(e')^{-1}(0)$ is open, there is an H , open in $\mathcal{G}_{(X, x_0)}(S, 0)$, with $U \in H$ and a neighbourhood V of x such that $(e')^{-1}(0) \supseteq H \times V$. Since $e'(W, y) = 0$ when $y \in W$, any element of H contains V . Now, consider an open cover of U , $\{U_i \mid i \in I\}$ and consider $U' = \bigcup \{U_i \mid i \in I\}$. Since $U' \supseteq U$, by Lemma 4a), $U' \in H$ and by Lemma 4b) there are $i_1, i_2, \dots, i_n \in I$ such that $U_{i_1} \cup U_{i_2} \cup \dots \cup U_{i_n} \in H$ and this implies that $U_{i_1} \cup U_{i_2} \cup \dots \cup U_{i_n} \supseteq V$.

Suppose now $x \notin \text{cl}\{x_0\}$. Therefore, there is an open A of X , such that $x \in A$ and $x_0 \notin A$. The open set $U \cap A \in \mathcal{G}_{(X, x_0)}(S, 1)$; the map $e: \mathcal{G}_{(X, x_0)}(S, 1) \wedge (X, x_0) \rightarrow (S, 1)$ is continuous and $e(U \cap A, x) = 0$. Replacing $U' = \bigcup \{U_i \mid i \in I\}$ by $U' = \bigcup \{U_i \cap A \mid i \in I\}$ in the argument used before proves that there exists a neighbourhood V of x with $U \cap A \supseteq V$, such that any open cover of U (and then of $U \cap A$), admits a finite subcover for V .

(2) $\rightarrow$ (3) Theorem 1.

(3) $\rightarrow$ (1) Trivial.

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REFERENCES

- [A] R. Arens, *A topology for spaces of transformations*, Ann. of Math. (2) **47** (1946), 480–495. MR **8**:165e
- [A-J] R. Arens and J. Dugundji, *Topologies for function spaces*, Pacific J. Math. **1** (1951), 5–31. MR **13**:264d
- [C-F] C. H. Cook and H. R. Fisher, *On equicontinuity and continuous convergence*, Math. Ann. **159** (1965), 94–104.
- [D] J. Dugundji, *Topology*, Allyn and Bacon, Boston, MA, 1966. MR **33**:1824
- [D-K] B. J. Day and G. M. Kelly, *On topological quotient maps preserved by pullbacks and products*, Math. Proc. Cambridge Philos. Soc. **67** (1970), 553–558. MR **40**:8024
- [F] R. H. Fox, *On topology for function-spaces*, Bull. Amer. Math. Soc. **51** (1945), 429–432. MR **6**:278d
- [H] D. Husemoller, *Fiber bundles*, McGraw-Hill, New York, 1966. MR **37**:4821
- [H-L] K. H. Hofmann and J. D. Lawson, *The spectral theory of distributive continuous lattices*, Trans. Amer. Math. Soc. **246** (1978), 285–310. MR **80c**:54045
- [J] I. M. James, *General topology and homotopy theory*, Springer-Verlag, Berlin and New York, 1984. MR **86d**:55001

- [I] J. R. Isbell, *Function spaces and adjoints*, Math. Scand. **36** (1975), 317–339. MR **53**:9134
- [Ma] S. Mac Lane, *Categories for the working mathematician* Graduate Texts in Math., vol. 5, Springer, Berlin and New York, 1971. MR **50**:7275
- [M] C. R. F. Maunder, *Algebraic topology*, Cambridge Univ. Press, Cambridge, 1980. MR **84k**:55001
- [W] S. Willard, *General topology*, Addison-Wesley, Reading, MA, 1970. MR **41**:9173

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